Axiom of Choice, Maximal Independent Sets, Argumentation and Dialogue Games*

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Abstract

In this work we investigate infinite structures. We discuss the importance, meaning and temptation of the axiom of choice and equivalent formulations with respect to graph theory, abstract argumentation and dialogue games. Emphasis is put on maximal independent sets in graph theory as well as preferred semantics in abstract argumentation.

1998 ACM Subject Classification G.2.2 Graph Theory, F.4.1 Mathematical Logic, I.2 Artificial Intelligence

Keywords and phrases axiom of choice, graph theory, maximal independent sets, abstract argumentation, dialogue games

Digital Object Identifier 10.4230/OASIcs.ICCSW.2014.91

1 Introduction

In everyday life we hardly ever think of dealing with actually infinite structures. Our time on earth may be complicated but it appears to be strictly finite, we deal with finite space, finite distances and finite numbers. Computer scientists in particular tend to prefer working with finite structures, algorithms are supposed to terminate in a finite amount of time. Often enough infinity introduces odd behaviour and exceptions to the languages we use, and grew to love. Therefore in everyday life infinity seems to be ignored.

However infinite structures actually are important even in our everyday life, see [18] for a fabulous overview in that matter. It might be pointed out that our concepts of economy and wealth build upon the idea of possibly infinite growth. Leibniz and Newton independently introduced infinitesimal calculus to a wider field of mathematics which nowadays plays major roles in almost every application of calculus to everyday life. On a more abstract level the failure of the halting problem is strongly related to a possible infinity. On the one hand it might seem disappointing that we will never be able to list all the perfect algorithms, on the other hand it comes as a relieve that we can always improve. Finally for security issues and problems, often enough specific finite limits play a major role.

If for one reason or another we agree to consider arbitrarily infinite structures then at some point we will also have to decide on what concept of infinity we take into account. In this paper we work with Zermelo-Fraenkel Set Theory [7, 14] and focus on the most controversial concept therein so far: The axiom of choice [13], the axiomatic existence of a choice function selecting exactly one element from an arbitrary set of arbitrary sets.

* This research has been supported by FWF (project I1102).

Imperial College Computing Student Workshop (ICCSW'14).
Editors: Rumyana Neykova and Nicholas Ng; pp. 91–98
OpenAccess Series in Informatics
OASICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Gödel proved consistency of choice with axiomatic set theory in [11]. However a few years later Cohen [6] showed that also its negation is relatively consistent. Which leaves us with its independence and thus the painful choice of accepting it or not.

Focus of this paper is on abstract argumentation as introduced by Dung in his seminal paper [8]. An abstract argumentation framework can be visualized as a graph where nodes reflect arguments and directed edges reflect conflicts between arguments. An argumentation semantics is a set of rules to declare sets of arguments as acceptable. Dialogue games are often motivated by real-life arguments and thus naturally play an important role in the motivation of some argumentation semantics.

In Section 2 we will introduce some basic concepts of set theory and infinity, and discuss the axiom of choice, alternate forms and motivation. In Section 3 we will proceed to discuss choice in the context of infinite graph theory [19]. It is known that the axiom of choice proves equivalent to the existence of a spanning tree for connected graphs and also to the existence of a maximal independent set [9]. We elaborate on the later result, and extend it to cover also preferred semantics of abstract argumentation [8] in Section 4. Finally in Section 5 we will introduce the axiom of determinacy, as used in dialogue games, we will discuss conflicts between the introduced axioms, draw connections and point out difficulties.

2 Set Theory and the Axiom of Choice

In this section we present notions and conventions. As mathematical basis we choose Zermelo-Fraenkel set theory [7, 14]. Observe that we make use of sets of sets, a differentiation between sets and elements by the use of uppercase and lowercase letters is therefore not possible. We will thus use the same lowercase letters to refer to sets as well as to elements.

▶ **Definition 1** (ZF-Axioms)**.** This is merely a selection of the axioms of interest. As an extended introduction into the matter we can recommend [7].

- Axiom schema of restricted comprehension (COMP): we can construct subsets of sets obeying some appropriate formula.
- Axiom of union (UN): The union over the elements of a set is a set.
- Axiom schema of replacement (REP): The image of a set under any definable function will also fall inside a set.
- Axiom of power set (POW): For any set x, there is a set y containing every subset of x.

While there are about (depending on actual definitions) nine axioms in ZF we only gave the above four, necessary here. Also some formalisms try to avoid one or the other of these axioms, as the less axioms a theory needs the stronger it becomes.

Axiomatic set theory proves especially helpful when speaking about infinity, as concepts that seem straightforward in the finite case often turn out to be of paradoxic nature in the infinite case. Since [10] we know that incompleteness necessarily is an inherent feature of reasonably strong formal systems, and thus independence [15] often enough the closest statement to consistency we can actually achieve.

▶ **Definition 2** (Choice)**.** We present the one axiom in many costumes that gave rise to the most objections, and is thus not a mandatory member of ZF. We choose two equivalent definitions. In the literature ZF together with any of these is called ZFC, Zermelo-Fraenkel set theory with choice.

- Axiom of Choice (AC): For any set x of nonempty sets, there exists a choice function f defined on x, i.e. $\forall y \in x : f(y) \in y$.
- Zorn's Lemma (ZL): In a partially ordered set where every chain has an upper bound there is at least one maximal element.

It is a well-known fact that in ZF we have that the extending assumptions (AC) and (ZL) are equivalent, in that each implies the other. When dealing with discrete structures (ZL) often comes in handy, (AC) on the other hand comes most often into play when there is no intuitive understanding of underlying structures. However in ZF these two (and countless others) can be used interchangeably. We would like to point out that countable choice (x contains only countably many sets) and also finite choice (x contains only finite sets) are substantially weaker than choice. An intuitive understanding of some structure might be misleading as uncountable structures seem to lie beyond human comprehension.

3 Graph Theory

Graph Theory [3, 23] lies at the core of discrete mathematics, with a wide range of applications such as shortest paths, and also with simple and clean definitions. Textbooks on algorithms vastly refer to some graph like structure as their standard model.

▶ **Definition 3** (Graph Theory). A *graph* G is a pair $G = (V, E)$ where the set of *vertices* V is an arbitrary set and the set of *edges* E is a collection of two-element subsets of V. For any two vertices $v_1, v_2 \in V$ with $\{v_1, v_2\} \in E$ we say that v_1 and v_2 are *adjacent*. If on the other hand $\{v_1, v_2\} \notin E$ we say that v_1 and v_2 are *independent*. A set of vertices $W \subseteq V$ is called *independent* iff it does not contain any adjacent vertices, in other words iff all vertices in W are pairwise independent. A set $W \subseteq V$ is called a *maximal independent set* of vertices in G iff it is independent and one of the following equivalent conditions holds:

- W is adjacent to any vertices not being member of W, i.e. for $v \in V \setminus W$ there is some $w \in W$ such that $\{v, w\} \in E$.
- W is maximal, i.e. there is no independent set W' such that $W \subsetneq W'$.

3.1 Choice in Graph Theory

We now introduce the statement about the existence of maximal independent sets, which will be shown to be another equivalent definition to (AC), the axiom of choice.[1]

▶ **Definition 4** (Existence of Maximal Independent Sets (MIS)).

- For any graph G there exists a maximal independent set M_G.

▶ **Theorem 5** ((AC) $\implies$ (MIS)). *Assuming Zermelo-Fraenkel Set Theory with Choice (ZFC) every graph has a maximal independent set.*

We take some graph $G = (V, E)$ as given. Now the objective is to construct some maximal independent set $M \subseteq V$ such that

1. M is independent, i.e. there are no $v, w \in M$ such that $\{v, w\} \in E$ and even
2. M is maximal independent, i.e. for any $v \in V \setminus M$ there is some $m \in M$ such that $\{v, m\} \in E$.

▶ Remark (Proof using (ZL)). We observe that sets of nodes represent a partially ordered set using the subset relation. Now for chains of independent sets $(M_i)_i$, where $M_i \subseteq M_j$ for $i \leq j$, we have that $\bigcup_i M_i$ is a natural upper bound for M_i. Using (ZL) we conclude that there has to be a maximum, i.e. a maximal independent set in G. We still present the following proof to highlight respective constructiveness.

[1] As pointed out by an anonymous reviewer this result is not exactly novel. In fact the proof for Theorem 2.4 in [9] follows the same ideas, although it uses an alternate form of (AC) and it provides one *not* too many, i.e. x, y should be adjacent iff they are R-equivalent.

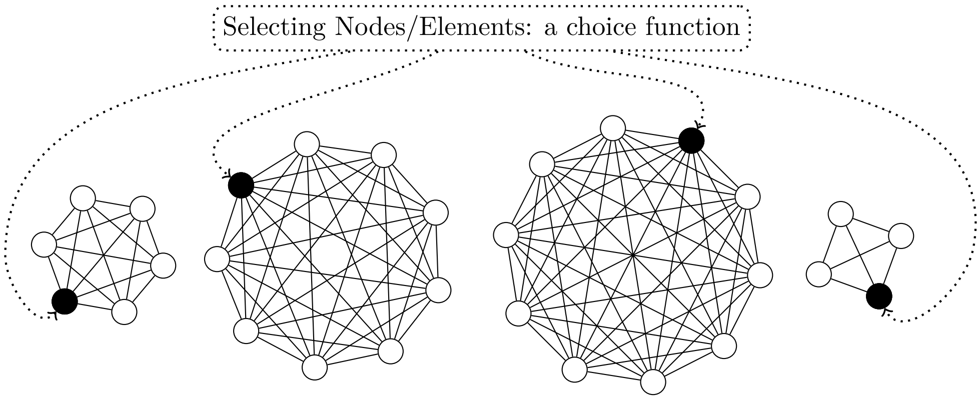

Figure 1 Illustration of the construction used for (MIS) $\Rightarrow$ (AC) as hinted to in Example 6.

▶ Remark (Cardinality and Ordinal Numbers). In axiomatic set theory ordinal numbers are an extension of natural numbers, a tool for counting infinities. For a given set x we call the smallest ordinal α for which there is a bijection between x and α its cardinality, $|x| = \alpha$. Observe that cardinality actually is strongly linked to (AC).

Proof. Observe that the empty set $\emptyset \subseteq V$ is an independent set of G. We use transfinite recursion to construct a maximal independent set and thus start with $M_0 = \emptyset$.

For any ordinal i either M_i is already maximal independent or there is some $v \in V \setminus M_i$ with $M_i \cup \{v\}$ still being independent, we then use $M_{i+1} = M_i \cup \{v\}$.

For limit ordinals α we use $M_\alpha = \bigcup_{i \in \alpha} M_i$. Observe that for this step we might need the choice function for the possibly infinitely many choices being necessary to receive M_α.

Since the Axiom of Choice holds and V is a set (using (AC) every set has a cardinality, i.e. transfinite recursion has to come to an end) this process will eventually stop, i.e. there is some $M = \bigcup_i M_i$. We have that $M \subseteq V$ and therefore M is a set. Furthermore since dependence is a finite condition M has to be independent. ◀

We can thus use any independent set and any node to start the construction of a maximal independent set. But what about the other direction. Assume (MIS), does this tell us something about (AC)?

▶ **Example 6** (Graph-Theoretical Motivation). We interpret a set of sets as an independent collection of cliques (subgraphs where any two vertices are adjacent). In other words for a given set of sets X we have that any $y \in Y \in X$ represents a distinct vertex and any two vertices are adjacent iff they originate from the same set $Y \in X$. Now (MIS) delivers a maximal independent set M where exactly one vertex for each clique and thus exactly one member for each set is fixed. An illustration of the resulting graph can be found in Figure 1.

▶ **Theorem 7** ((MIS) $\Longrightarrow$ (AC)). *Assuming Zermelo-Fraenkel Set Theory without Choice (ZF) and the existence of a maximal independent set for arbitrary graphs (MIS) we can derive the axiom of choice (AC).*

We take some set of nonempty sets X as given. Now the objective is to find a choice function $f : X \to \bigcup X$ such that for any set $Y \in X$ we have $f(Y) \in Y$. Observe that in mathematical terms a function actually is just a specific kind of relation and can thus be defined as a set of pairs.

Proof. We start by constructing a graph $G = (V, E)$, where V consists of all the pairs (Y, y) such that $Y \in X$ and $y \in Y$, and adjacency in E is defined by belonging to the same set Y and being different, i.e. $\{(Y_1, y_1), (Y_2, y_2)\} \in E$ iff $Y_1 = Y_2$ and $y_1 \neq y_2$.

The above steps in more detail:

- We use (REP) to create a set X' such that $Y' \in X'$ iff any $y' \in Y'$ is of the form (Y, y) where $Y \in X$ and $y \in Y$.
- We use (UN) to create $V = \bigcup Y'$.
- We use (REP), (POW), (COMP) and (UN) to create E. I.e. E is constructed by replacing each member of Y' with its size two suitable subsets $\{\{x, y\} \,|x \neq y \in Y'\} \subseteq \mathcal{P}(Y')$ and finally we collect all of those in one single set E by using (UN).

Now assuming (MIS) we conclude that G provides a maximal independent set $M \subseteq V$. By construction M consists of pairs of the form (Y, y) where $y \in Y \in X$. It remains to show that M already serves the purpose of being a choice function for X, mapping Y to y.

We observe that for $v = (Y_1, y_1)$ and $w = (Y_2, y_2)$ such that $v \neq w \in V$ we have v being adjacent to w iff $Y_1 = Y_2$ and v being independent from w iff $Y_1 \neq Y_2$.

M is independent. For each $Y \in X$ there is thus at most one $y \in Y$ such that $(Y, y) \in M$, since M is maximal independent there is furthermore at least and thus exactly one $y \in Y$ such that $(Y, y) \in M$. ◀

▶ Remark (Directed Graphs). A *directed graph* is a pair $D = (V, E)$, where V is an arbitrary set of vertices but this time the set of edges $E \subseteq V \times V$ is a set of pairs. Observe that this allows edges also from one node to itself. The definition of independent sets and proofs for equivalence with axiom of choice carry on also for the directed case.

4 Abstract Argumentation

Abstract argumentation can be seen as applied directed graph theory. It was introduced by Dung in [8] and motivated by philosophical works [12, 20] and non-monotonic logic [4]. In this work we also want to highlight the interplay of argumentation and dialogue games [21].

▶ **Definition 8.** An *argumentation framework (AF)* is an ordered pair $F = (A, R)$ where A is an arbitrary set of *arguments* and $R \subseteq A \times A$ is called the *attack* or *conflict* relation. For $(a, b) \in R$ we also write $a \rightarrowtail^R b$ and say that a attacks b in R. Furthermore for $(a, b), (b, c) \in R$ we say that a defends c against b in R.

If clear from context we might omit the referencing R or other redundant information. If not otherwise stated we will also assume some AF $F = (A, R)$ as given.

Furthermore for $B \subseteq A$ and $a \in A$ we say that $a \rightarrowtail B$ or $B \rightarrowtail a$ iff for some $b \in B$ we have $a \rightarrowtail b$ or $b \rightarrowtail a$ respectively. We extend this notion also for $B, C \subseteq A$ accordingly.

▶ Remark (Directed Graphs and Argumentation Frameworks). Observe that so far abstract argumentation frameworks and directed graphs are formally equivalent, except for names. The main difference being one of intended meaning. In graph theory we think of nodes being near to each other if they are connected by an edge, or even connected via a path of several nodes and edges. Furthermore if two nodes do not cooperate we could reflect this only indirectly by not having any connection between them. In argumentation theory on the other hand nearness is expressed only indirectly via the notion of defense, while an attack represents a conflict between two arguments.

What follows is the notion of semantics as used in abstract argumentation, an attempt of solving the question of acceptability. Investigating some arbitrary AF we might want to consider some sets of arguments, for instance an argument line of defense. A good line of defense of course is a line of attack. Truth in the argumentation sense is thus a notion of acceptability. For a comprehensive introduction into argumentation semantics see [1].

▶ **Definition 9.** An *argumentation semantics* is a mapping from AFs to sets of arguments where for any AF $F = (A, R)$ and semantics σ we have $\sigma(F) \subseteq \mathcal{P}(A)$. The members of $\sigma(F)$ are then called *σ-extensions* of F. The other way around we will define semantics by stating properties an extension has to fulfill. We sometimes call an argument that is member of every σ-extension *sceptically accepted* and an argument that is member of some σ-extension *credulously accepted*.

A set of arguments $E \subseteq A$ is called

- *conflict-free* (*cf*) or a conflict-free extension iff there is no conflict in E, $(E \times E) \cap R = \emptyset$,
- *admissible* (*adm*) or an admissible extension iff it is *cf* and defends itself against any attacks, $E \in cf(F)$ and for any $a \rightarrowtail E$ we also have $E \rightarrowtail a$,
- a *preferred* (*prf*) extension iff it is maximal admissible, $E \in adm(F)$ and there is no $E' \in adm(F)$ with $E \subsetneq E'$.

4.1 Choice in Argumentation

The remainder of this section is a brief summary of implied (AC)-results for abstract argumentation. The first part of this theorem has already been discussed in the literature (see [8, 5]) although with use of (ZL). The second part however is a novel result and might still come as a surprise to some. Preferred Extensions are one of the most basic concepts in abstract argumentation and the necessary definitions seem to be rather intuitive and straightforward. It is not obvious that this concept of discrete structures carries over to e.g. existence of a base in the vector space of all continuous functions.

▶ **Definition 10** (Existence of Preferred Extensions (PE)).

- For any AF F there exists a preferred extension $prf(F) \neq \emptyset$.

▶ **Theorem 11** ((AC) $\iff$ (PE)). *In ZF, the axiom of choice is equivalent to the existence of preferred extensions in arbitrary AFs.*

Proof Part 1, $\Longrightarrow$: Very similar to (MIS) we use transfinite recursion and observe that the empty set is an admissible extension. If some admissible set is not maximal then there is a bigger one. And finally for any chain of admissible sets $E_1 \subsetneq E_2 \subsetneq \dots$ it holds that $E = \bigcup_i E_i$ also is an admissible set. Conflict is a finite condition and any argument $a \in E$ that is not defended by E was first introduced for some E_i and thus already E_i would not have been able to defend a. ◀

Proof Part 2, $\Longleftarrow$: Observe that the construction from the proof for Theorem 7 can be seen as an argumentation framework by the natural transformation from undirected to directed graphs, where for the graph $G = (V, E)$ we use an AF $F = (V, E')$ where $E' = \{(a, b) \mid \{a, b\} \in E\}$. Then obviously X is a preferred extension of F iff X is a maximal independent set of G iff X is a choice function for the originating set of sets. ◀

5 Discussion and Dialogue Games

When thinking about AFs we think about some mode of argumentation, with every argument being some sort of statement and the conflict relation being naturally induced by the origin of the arguments. We might acquire arguments (and implicit conflicts) from logical formulas or programs or from some natural language dialogue. A major source of motivation for abstract argumentation also are dialogue games [16, 22].

From the very beginning of argumentation it seems that dialogue games played an important motivational role, see [2, 8]. For a nice introduction into the use of dialogue games to reason about credulous and skeptical preferred acceptability also see [21]. We sometimes think of argumentation as an interpretation of dialogue games on an attack graph. A move can be seen as the act of claiming an argument, where Opponent selects attacking arguments and Proponent selects defending arguments. Observe that credulous acceptance of an argument for preferred semantics can be implemented with the idea of Proponent indefinitely being able to defend his selected arguments.

▶ **Definition 12** (Dialogue Games)**.** A *dialogue game* is a two-player game, with alternating moves. In the case of *perfect information* we assume that consequences of any move are known in advance by both players and that both players are aware of all possible states. A *winning strategy* for one of the players is a function mapping game states into a set of moves such that the resulting sequence indicates a win by the respective player. We call the player to start the game *Proponent* and the player to answer first *Opponent*.

When digging into game theory and reflecting the axiom of choice one sooner or later stumbles upon determinacy, the question of whether winning strategies do exist and whether a game is decided before it actually started. For games on fixed AFs we would expect determinacy and we would also hope for a one-to-one relation between abstract argumentation and dialogue games. However sometimes what we hope for is not what we get.

▶ **Example 13** (The number game $\mathbb{S}$)**.** Take some arbitrary set $\mathbb{S} \subseteq \mathbb{N}^{\mathbb{N}}$. We define a dialogue game of length $|\mathbb{N}|$ where moves are natural numbers, i.e. possible game sequences are of the form $(n_1, n_2, \cdots)$. Proponent wins iff the played sequence is an element of $\mathbb{S}$.

▶ **Definition 14** (Axiom of Determinacy (AD))**.**

- Every number game $\mathbb{S}$ is predetermined, i.e. one of the players has a winning strategy.

We now want to highlight a quite interesting result [17]: the axioms of choice and of determinacy are incompatible, i.e. (AC) implies not (AD), in other words choice attacks determinacy. The proof idea here is a classical diagonal argument. Now take into account the above result that existence of a preferred extension and the axiom of choice are equivalent. It follows that the idea of determinacy, allowing winning strategies in perfect information games and existence of preferred extensions for arbitrary AFs are incompatible as well.

Once more we point out that most of computer science takes place in the finite or countably infinite case. As (AD) implies countable choice the stated conflict in most cases will not be of immediate relevance. From time to time we should just remind ourselves that a vast generalization of techniques and results might not be possible. In this context the really interesting question will be which axiom to trust in the arbitrarily infinite case. Do we prefer every graph to contain a maximal clique and every argumentation framework to contain at least one preferred extension, or do we prefer every dialogue game with perfect information to be predetermined? What other results can we gain distinguishing these cases? Are there for instance argumentation semantics that rely on existence of winning strategies? For which AFs and which games can we find transformations/mappings such that winning strategies and some semantics correlate?

Acknowledgements. We would like to thank the anonymous reviewers for their helpful and constructive comments.

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