Complexity Theory

VU 181.142, WS 2019

5. NP-Completeness

Reinhard Pichler

Institut für Informationssysteme
Arbeitsbereich DBAI
Technische Universität Wien

4 November, 2019

Outline

Some Variants of Satisfiability

We have already encountered several versions of satisfiability problems:

- intractable: **SAT**, **3-SAT**
- tractable: **2-SAT**, **HORNSAT**

We shall encounter further intractable versions of satisfiability problems:

- restricted (but still intractable) versions of **SAT**
- **CIRCUIT SAT**
- **Not-all-equal SAT** (**NAESAT**)
- **(MONOTONE) 1-IN-3-SAT**
- strongly related problem: **HITTING SET**

Narrowing NP-complete languages

An NP-complete language can sometimes be narrowed down by transformations which eliminate certain features of the language but still preserve NP-completeness.

Restricting **SAT** to formulae in CNF and a further restriction to **3-SAT** are typical examples. Generally, **k-SAT** (i.e., formulae are restricted to CNF with exactly k literals in each clause) is NP-complete for any $k \geq 3$.

Here is another example of narrowing an NP-complete language:

Proposition

3-SAT *remains* NP-*complete even if the Boolean expressions* φ *in 3-CNF are restricted such that*

- *each variable appears at most three times in* φ *and*
- *each literal appears at most twice in* φ*.*

Proof

The reduction consists in rewriting an arbitrary instance φ of **3-SAT** in such a way that the forbidden features are eliminated.

Consider a variable x appearing $k > 3$ times in φ.

(i) Replace the first occurrence of x in φ by x_1, the second by x_2, and so on where $x_1, \ldots, x_k$ are new variables.

(ii) Add clauses $(\neg x_1 \vee x_2), (\neg x_2 \vee x_3), \ldots, (\neg x_k \vee x_1)$ to φ.

Let φ' be the result of systematically modifying φ in this way. Clearly, φ' has the desired syntactic properties.

Now φ is satisfiable iff φ' is satisfiable:
For each x appearing $k > 3$ times in φ, the truth values of $x_1, \ldots, x_k$ are the same in each truth assignment satisfying φ'.

Boolean Circuits

Syntax of Boolean circuits

- A Boolean circuit is a directed graph $C = (V, E)$ where $V = \{1, 2, \ldots, n\}$ is the set of gates and C is acyclic (with $i < j$ for all edges $(i, j) \in E$).
- All gates i have a sort $s(i) \in \{\mathbf{true}, \mathbf{false}, \wedge, \vee, \neg\} \cup \{x_1, x_2, \ldots\}$.
  – If $s(i) \in \{\mathbf{true}, \mathbf{false}\} \cup \{x_1, x_2, \ldots\}$, the indegree of i is 0 (inputs).
  – If $s(i) = \neg$ then the indegree of i is 1.
  – If $s(i) \in \{\vee, \wedge\}$ then the indegree of i is 2.
- Gate n is the output of the circuit.

Remark. $\{x_1, x_2, \ldots\}$ are variables whose value can be **true** or **false**.

Boolean Circuits

Semantics

Let C be a Boolean circuit and let $X(C)$ denote the set of variables appearing in the circuit C. A truth assignment for C is a function $T : X(C) \to \{\mathbf{true}, \mathbf{false}\}$.

The truth value $T(i)$ for each gate i is defined inductively:

- If $s(i) = \mathbf{true}$, $T(i) = \mathbf{true}$ and if $s(i) = \mathbf{false}$, $T(i) = \mathbf{false}$.
- If $s(i) = x_j \in X(C)$, then $T(i) = T(x_j)$.
- If $s(i) = \neg$, then $T(i) = \mathbf{true}$ if $T(j) = \mathbf{false}$, else $T(i) = \mathbf{false}$ where (j, i) is the unique edge entering i.
- If $s(i) = \wedge$, then $T(i) = \mathbf{true}$ if $T(j) = T(j') = \mathbf{true}$ else $T(i) = \mathbf{false}$ where (j, i) and (j', i) are the two edges entering i.
- If $s(i) = \vee$, then $T(i) = \mathbf{true}$ if $T(j) = \mathbf{true}$ or $T(j') = \mathbf{true}$ else $T(i) = \mathbf{false}$ where (j, i) and (j', i) are the two edges entering i.
- $T(C) = T(n)$, i.e. the value of the circuit C.

CIRCUIT SAT

CIRCUIT SAT

INSTANCE: Boolean circuit C with variables $X(C)$

QUESTION: Does there exist a truth assignment $T : X(C) \to \{\mathbf{true}, \mathbf{false}\}$ such that $T(C) = \mathbf{true}$?

Theorem

CIRCUIT SAT *is* NP*-complete.*

Proof of NP-Membership

Consider the following NP-algorithm:

1. Guess a truth assignment $T : X(C) \to \{\mathbf{true}, \mathbf{false}\}$.
2. Check that $T(C) = \mathbf{true}$ holds.

Proof of NP-Hardness

We prove the NP-hardness by a reduction from **SAT**: Let an arbitrary instance of **SAT** be given by a Boolean formula φ over the variables $X = \{x_1, \ldots, x_k\}$. We construct the following Boolean circuit $C(\varphi)$:

- The variables $X(C)$ in $C(\varphi)$ are precisely the variables X.
- For every subexpression ψ of φ, $C(\varphi)$ contains a gate $g(\psi)$. The output gate of $C(\varphi)$ is the gate $g(\varphi)$.
- The sort and the incoming arcs of each gate $g(\psi)$ in $C(\varphi)$ are defined inductively:
  - If ψ is a variable x_i then $g(\psi)$ is an input gate of sort $s(g(\psi)) = x_i$
  - If $\psi = \neg\psi'$ then $s(g(\psi)) = \neg$ with an incoming arc from $g(\psi')$.
  - If $\psi = \psi_1 \wedge \psi_2$ (resp. $\psi = \psi_1 \vee \psi_2$), then $s(g(\psi)) = \wedge$ (resp. $s(g(\psi)) = \vee$) with incoming arcs from $g(\psi_1)$ and $g(\psi_2)$.

Reduction from **SAT** to **3-SAT**

Motivation

- We have already seen how an arbitrary propositional formula φ can be transformed efficiently into a sat-equivalent formula ψ in 3-CNF.
- This transformation (first into CNF and then into 3-CNF) is intuitive and clearly works in polynomial time. However, the log-space complexity of this transformation is not immediate.
- We now give an alternative transformation by reducing **CIRCUIT SAT** to **3-SAT**. In total, we thus have:

$$\textbf{SAT} \leq_{\mathrm{L}} \textbf{CIRCUIT SAT} \leq_{\mathrm{L}} \textbf{3-SAT}$$

Reduction from **CIRCUIT SAT** to **3-SAT**

Let an arbitrary instance of **CIRCUIT SAT** be given by a Boolean circuit C. We construct the following instance $\varphi(C)$ of **SAT** (φ is in CNF with some clauses smaller than 3. The transformation into 3-CNF is obvious):

The formula $\varphi(C)$ uses all variables of C. Moreover, for each gate g of C, $\varphi(C)$ has a new variable g and the following clauses.

1. If g is a variable gate x: $(g \vee \neg x), (\neg g \vee x)$. $[g \leftrightarrow x]$
2. If g is a **true** (resp. **false**) gate: g (resp. $\neg g$).
3. If g is a NOT gate with a predecessor h:
   $(\neg g \vee \neg h), (g \vee h)$. $[g \leftrightarrow \neg h]$
4. If g is an AND gate with predecessors h, h':
   $(\neg g \vee h), (\neg g \vee h'), (g \vee \neg h \vee \neg h')$. $[g \leftrightarrow (h \wedge h')]$
5. If g is an OR gate with predecessors h, h':
   $(\neg g \vee h \vee h'), (g \vee \neg h'), (g \vee \neg h)$. $[g \leftrightarrow (h \vee h')]$
6. If g is also the output gate: g.

NAESAT

Not-all-equal SAT (NAESAT)

INSTANCE: Boolean formula φ in 3-CNF

QUESTION: Does there exist a truth assignment T appropriate to φ, such that the 3 literals in each clause do not have the same truth value?

Remark. Clearly **NAESAT** $\subset$ **3-SAT**.

Theorem

NAESAT *is* NP*-complete.*

NAESAT

Proof of NP-Hardness

Recall the Boolean formula $\varphi(C)$ resulting from the reduction of **CIRCUIT SAT** to **3-SAT**. For all one- and two-literal clauses in the resulting CNF-formula $\varphi(C)$, we add the same literal z (possibly twice) to make them 3-literal clauses.

The resulting formula $\varphi_z(C)$ fulfills the following equivalence:

$$\varphi_z(C) \in \textbf{NAESAT} \Leftrightarrow C \in \textbf{CIRCUIT SAT}.$$

"$\Rightarrow$" If a truth assignment T satisfies $\varphi_z(C)$ in the sense of **NAESAT**, so does the complementary truth assignment $\overline{T}$.
Thus, z is **false** in either T or $\overline{T}$ which implies that $\varphi(C)$ is satisfied by either T or $\overline{T}$. Thus C is satisfiable.

NAESAT

Proof of NP-Hardness (continued)

"$\Leftarrow$" If C is satisfiable, then there is a truth assignment T satisfying $\varphi(C)$. Let us then extend T for $\varphi_z(C)$ by assigning $T(z) =$ **false**.

By assumption, T is a satisfying truth assignment of $\varphi(C)$ and, therefore, also of $\varphi_z(C)$. Hence, in no clause of $\varphi_z(C)$ all literals are **false**.
It remains to show that in no clause of $\varphi_z(C)$ all literals are **true**:

(i) Clauses for **true**/**false**/NOT/variable gates contain z that is **false**.

(ii) For AND gates the clauses are: $(\neg g \vee h \vee z)$, $(\neg g \vee h' \vee z)$, $(g \vee \neg h \vee \neg h')$ where in the first two z is **false**, and in the third all three cannot be **true** as then the first two clauses would be **false**.

(iii) For OR gates the clauses are: $(\neg g \vee h \vee h'), (g \vee \neg h' \vee z)$, $(g \vee \neg h \vee z)$ where in the last two z is **false**, and in the first all three cannot be **true** as then the last two clauses would be **false**.

1-IN-3-SAT

1-IN-3-SAT

INSTANCE: Boolean formula φ in 3-CNF

QUESTION: Does there exist a truth assignment T appropriate to φ, such that in each clause, exactly one literal is **true** in T?

MONOTONE 1-IN-3-SAT

INSTANCE: Boolean formula φ in 3-CNF, s.t. the clauses in φ contain only unnegated atoms.

QUESTION: Does there exist a truth assignment T appropriate to φ, such that in each clause, exactly one literal is **true** in T?

Theorem

Both **1-IN-3-SAT** *and* **MONOTONE 1-IN-3-SAT** *are* NP*-complete.*

1-IN-3-SAT

Remarks

- Clearly **1-IN-3-SAT** $\subset$ **NAESAT** $\subset$ **3-SAT**. The instances of these 3 problems are the same, namely 3-CNF formulae. However, the positive instances of **1-IN-3-SAT** are a proper subset of **NAESAT**, which in turn are a proper subset of the positive instances of **3-SAT**.
- Note that the NP-completeness of any of these 3 problems does not immediatetely imply the NP-completeness of any of the other problems, since it is a priori not clear if further constraining the positive instances makes things easier or harder.
- **MONOTONE 1-IN-3-SAT** is a special case of **1-IN-3-SAT**, i.e., the instances of the former are a proper subset of the latter while the question remains the same. The NP-hardness of the special case immediately implies the NP-hardness of the general case.

Proof of the NP-hardness of **1-IN-3-SAT**

We prove the NP-hardnes by a reduction from **4-SAT**:
Let φ be an arbitrary instance of **4-SAT**, i.e., φ is in 4-CNF.
We construct an instance ψ of **1-IN-3-SAT** as follows:

For every clause $l_1 \vee l_2 \vee l_3 \vee l_4$ in φ, let $a_1, a_2, a_3, a_4, b_1, b_2, c_1, c_2, d$ be 9 fresh propositional variables. Then ψ contains the following 7 clauses:

(1) $l_1 \vee a_1 \vee b_1$
(2) $l_2 \vee a_2 \vee b_1$
(3) $a_1 \vee a_2 \vee c_1$
(4) $l_3 \vee a_3 \vee b_2$
(5) $l_4 \vee a_4 \vee b_2$
(6) $a_3 \vee a_4 \vee c_2$
(7) $b_1 \vee b_2 \vee d$

Idea. These seven clauses guarantee that in a legal 1-in-3 assignment of ψ, the clause $l_1 \vee \cdots \vee l_4$ must be **true**:
By (1) – (3): If l_1 and l_2 are **false**, then b_1 must be **true**.
By (4) – (6): If l_3 and l_4 are **false**, then b_2 must be **true**.
However, by (7), it is not allowed that both b_1 and b_2 are **true**.

Proof of the NP-hardness of **MONOTONE 1-IN-3-SAT**

We show how an arbitrary instance φ of **1-IN-3-SAT** can be transformed into an equivalent instance ψ of **MONOTONE 1-IN-3-SAT**:

Let $X = \{x_1, \ldots, x_n\}$ be the variables in φ. Then the variables in ψ are $X \cup \{x_i' \mid 1 \leq i \leq n\} \cup \{a, b, c\}$. In φ, we replace every negative literal of the form $\neg x_i$ (for some i) by the unnegated atom x_i'.

Moreover, for every $i \in \{1, \ldots, n\}$, we add the following 3 clauses:

(1) $x_i \vee x_i' \vee a$
(2) $x_i \vee x_i' \vee b$
(3) $a \vee b \vee c$

Idea. These three clauses guarantee that in a legal 1-in-3 assignment of ψ, the variables x_i and x_i' have complementary truth values. Hence, x_i' indeed encodes $\neg x_i$.

HITTING SET

HITTING SET

INSTANCE: Set $T = \{t_1, \ldots, t_p\}$, family $(V_i)_{1 \leq i \leq n}$ of subsets of T, i.e.: for all $i \in \{1, \ldots, n\}$, $V_i \subseteq T$.

QUESTION: Does there exist a set $W \subseteq T$, s.t. $|W \cap V_i| = 1$ for all $i \in \{1, \ldots, n\}$? (A set W with this property is called a "hitting set").

Corollary

HITTING SET *is* NP*-complete.*

Proof of the NP-hardness

By reduction from **MONOTONE 1-IN-3-SAT**: Let an instance of **MONOTONE 1-IN-3-SAT** be given by the 3-CNF formula φ over the variables X. We define the following instance of **HITTING SET**:

$T = X$. Moreover, suppose that φ contains n clauses. Then there are n sets $(V_i)_{1 \leq i \leq n}$. If the i-th clause in φ is $l_1 \vee l_2 \vee l_3$, then $V_i = \{l_1, l_2, l_3\}$.

Some Graph Problems

In the "Formal Methods in Computer Science" lecture, we have already proved the NP-completeness of the following graph problems:

- **INDEPENDENT SET**
- **CLIQUE**
- **VERTEX COVER**

We shall now show the following results:

- **3-COLORABILITY** is NP-complete.
- **HAMILTON-PATH** $\leq_{\mathrm{L}}$ **HAMILTON-CYCLE** $\leq_{\mathrm{L}}$ **TSP(D)**

INDEPENDENT SET

INSTANCE: Undirected graph $G = (V, E)$ and integer K.

QUESTION: Does there exist an *independent set* I of size $\geq K$?
i.e., $I \subseteq V$, s.t. for all $i, j \in I$ with $i \neq j$, $[i, j] \notin E$.

CLIQUE

INSTANCE: Undirected graph $G = (V, E)$ and integer K.

QUESTION: Does there exist a *clique* C of size $\geq K$?
i.e., $C \subseteq V$, s.t. for all $i, j \in C$ with $i \neq j$, $[i, j] \in E$.

VERTEX COVER

INSTANCE: Undirected graph $G = (V, E)$ and integer K.

QUESTION: Does there exist a *vertex cover* N of size $\leq K$?
i.e., $N \subseteq V$, s.t. for all $[i, j] \in E$, either $i \in N$ or $j \in N$.

Decision Problems

3-COLORABILITY

INSTANCE: Undirected graph $G = (V, E)$

QUESTION: Does G have a *3-coloring*? i.e., an assignment of one of 3 colors to each of the vertices in V such that any two vertices i, j connected by an edge $[i, j] \in E$ do not have the same color?

k-COLORABILITY (for fixed value k)

INSTANCE: Undirected graph $G = (V, E)$

QUESTION: Does G have a *k-coloring*? i.e., an assignment of one of k colors to each of the vertices in V such that any two vertices i, j connected by an edge $[i, j] \in E$ do not have the same color?

Complexity

Theorem

The **k-COLORABILITY***-problem is* NP*-complete for any fixed* $k \geq 3$*.*
The **2-COLORABILITY***-problem is in* P*.*

Proof

NP-Membership of **k-COLORABILITY**:
1. Guess an assignment $f : V \to \{1, \ldots, k\}$
2. Check for every edge $[i, j] \in E$ that $f(i) \neq f(j)$.

P-Membership of **2-COLORABILITY**: (w.l.o.g., G is connected)
1. Start by assigning an arbitrary color to an arbitrary vertex $v \in V$.
2. Suppose that the vertices in $S \subset V$ have already been assigned a color. Choose $x \in S$ and assign to all vertices adjacent to x the opposite color.

G is 2-colorable iff step 2 never leads to a contradiction.

NP-Hardness Proof of **3-COLORABILITY**

By reduction from **NAESAT**: Let an arbitrary instance of **NAESAT** be given by a Boolean formula $\varphi = c_1 \wedge \ldots \wedge c_m$ in 3-CNF with variables $x_1, \ldots, x_n$. We construct the following graph $G(\varphi)$:

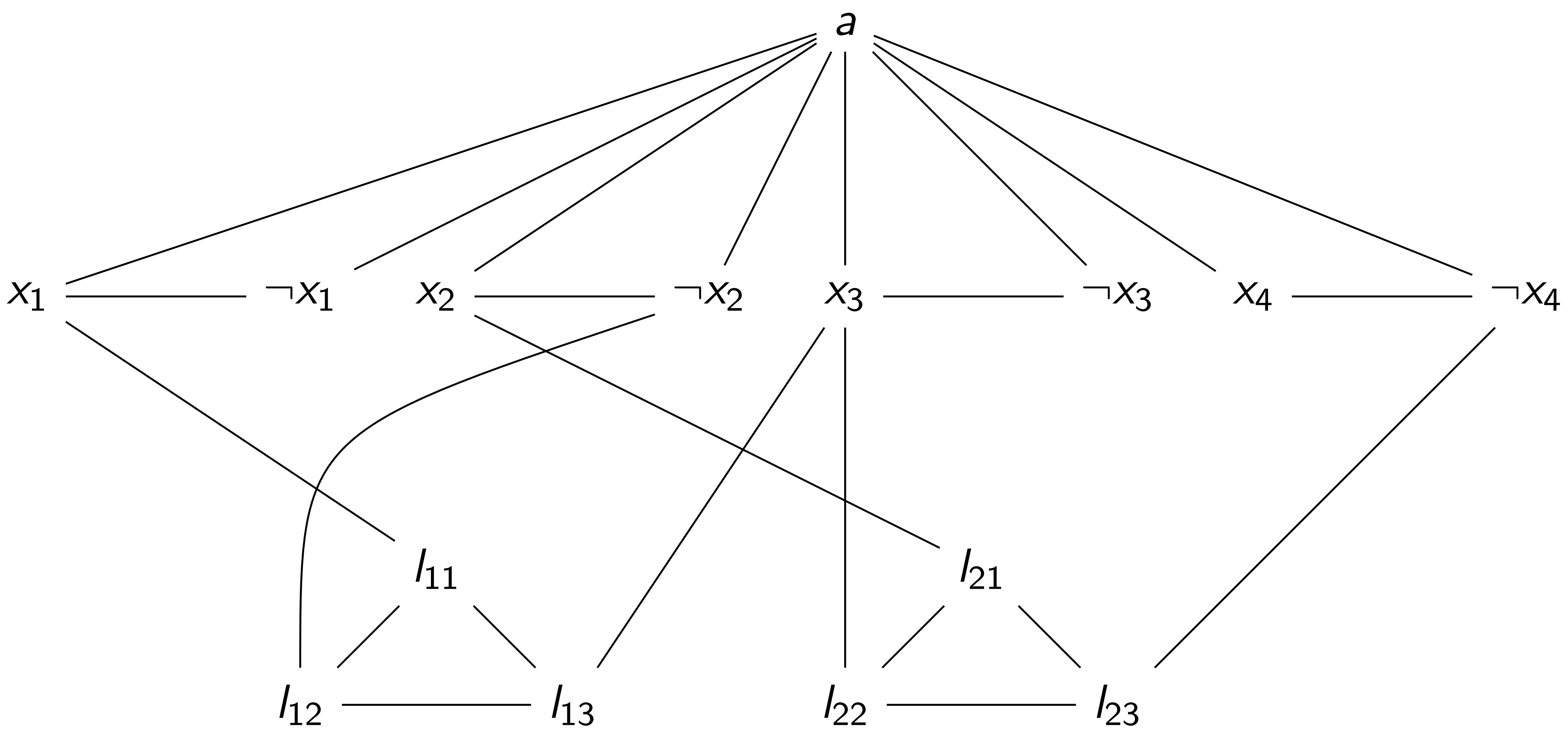
Let $V = \{a\} \cup \{x_i, \neg x_i \mid 1 \leq i \leq n\} \cup \{l_{i1}, l_{i2}, l_{i3} \mid 1 \leq i \leq m\}$,
i.e. $|V| = 1 + 2n + 3m$.

For each variable x_i in φ, we introduce a triangle $[a, x_i, \neg x_i]$,
i.e. all these triangles share the node a.

For each clause c_i in φ, we introduce a triangle $[l_{i1}, l_{i2}, l_{i3}]$. Moreover, each of these vertices l_{ij} is further connected to the node corresponding to this literal, i.e.: if the j-th literal in c_i is of the form x_α (resp. $\neg x_\alpha$) then we introduce an edge between l_{ij} and x_α (resp. $\neg x_\alpha$)

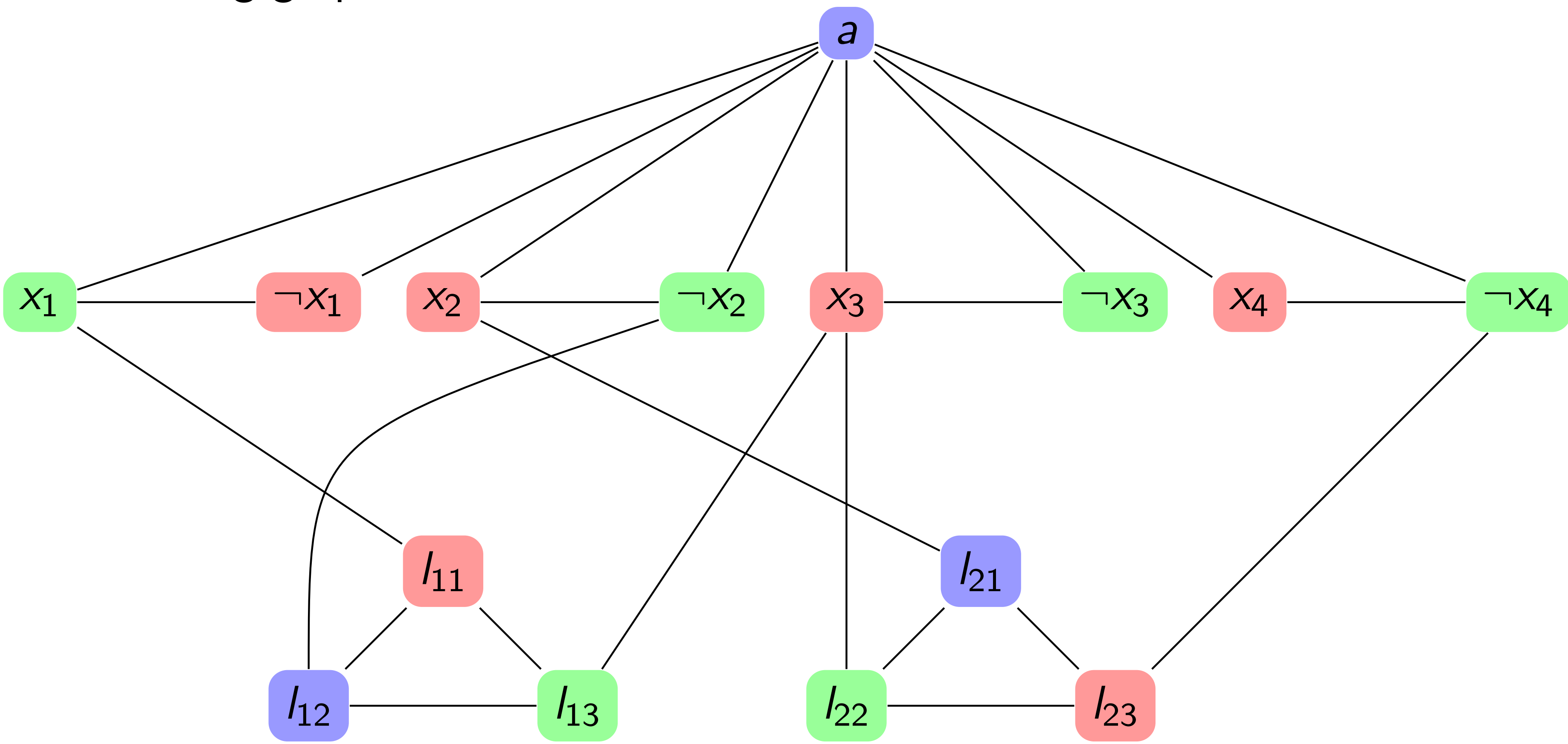
Example

The 3-CNF formula $\varphi = (x_1 \lor \neg x_2 \lor x_3) \land (x_2 \lor x_3 \lor \neg x_4)$ is reduced to the following graph:



Example

The 3-CNF formula $\varphi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee x_3 \vee \neg x_4)$ is reduced to the following graph:



Let red $=$ **false** and green $=$ **true**. The above 3-coloring corresponds to $T(x_1) = T(\neg x_2) = T(\neg x_3) = T(\neg x_4) =$ **true**.

Correctness of the Problem Reduction

Proof (continued)

"$\Leftarrow$" Suppose that G has a 3-coloring with colors $\{0, 1, 2\}$. W.l.o.g., the node a has the color 2. This induces a truth assignment T via the colors of the nodes x_i: if the color is 1, then $T(x_i) = \textbf{true}$ else $T(x_i) = \textbf{false}$. We claim that T is a legal **NAESAT**-assignment. Indeed, if in some clause, all literals had the value **false** (resp. **true**), then we could not use the color 0 (resp. 1) for coloring the triangle $[l_{i1}, l_{i2}, l_{i3}]$, a contradiction.

"$\Rightarrow$" Suppose that there exists an **NAESAT**-assignment T of φ. Then we can extract a 3-coloring for G from T as follows:

(i) Node a is colored with color 2.

(ii) If $T(x_i) = \textbf{true}$, then color x_i with 1 and $\neg x_i$ with 0 else vice versa.

(iii) From each $[l_{i1}, l_{i2}, l_{i3}]$, color two literals having opposite truth values with 0 (**true**) and 1 (**false**). Color the third with 2.

HAMILTON-PATH

INSTANCE: (directed or undirected) graph $G = (V, E)$

QUESTION: Does G have a *Hamilton path*?
i.e., a path visiting all vertices of G exactly once.

HAMILTON-CYCLE

INSTANCE: (directed or undirected) graph $G = (V, E)$

QUESTION: Does G have a *Hamilton cycle*?
i.e., a cycle visiting all vertices of G exactly once.

TSP(D)

INSTANCE: n cities $1, \ldots, n$ and a nonnegative integer distance d_{ij} between any two cities i and j (such that $d_{ij} = d_{ji}$), and an integer B.

QUESTION: Is there a tour through all cities of length at most B?
i.e., a permutation π s.t. $\sum_{i=1}^{n} d_{\pi(i)\pi(i+1)} \leq B$ with $\pi(n+1) = \pi(1)$.

Complexity

Theorem

HAMILTON-PATH, **HAMILTON-CYCLE**, *and* **TSP(D)** *are* NP*-complete.*

Proof

We shall show the following chain of reductions:

$$\textbf{HAMILTON-PATH} \leq_{\mathrm{L}} \textbf{HAMILTON-CYCLE} \leq_{\mathrm{L}} \textbf{TSP(D)}$$

It suffices to show NP-membership for the *hardest* problem:

1. Guess a tour π through the n cities.
2. Check that $\sum_{i=1}^{n} d_{\pi(i)\pi(i+1)} \leq B$ with $\pi(n+1) = \pi(1)$.

Likewise, it suffices to prove the NP-hardness of the *easiest* problem.
The NP-hardness of **HAMILTON-PATH** (by a reduction from **3-SAT**) is quite involved and is therefore omitted here (see Papadimitriou's book).

HAMILTON-PATH vs. HAMILTON-CYCLE

HAMILTON-PATH $\leq_{\mathrm{L}}$ HAMILTON-CYCLE

(We only consider undirected graphs). Let an arbitrary instance of **HAMILTON-PATH** be given by the graph $G = (V, E)$. We construct an equivalent instance $G' = (V', E')$ of **HAMILTON-CYCLE** as follows:

Let $V' := V \cup \{z\}$ for some new vertex z and $E' := E \cup \{[v, z] \mid v \in V\}$.
G has a Hamilton path $\Leftrightarrow$ G' has a Hamilton cycle

"$\Rightarrow$" Suppose that G has a Hamilton path π starting at vertex a and ending at b. Then $\pi \cup \{z\}$ is clearly a Hamilton cycle in G'.

"$\Leftarrow$" Let C be a Hamilton cycle in G'. In particular, C goes through z. Let a and b be the two neighboring nodes of z in this cycle. Then $C \setminus \{z\}$ is a Hamilton path (starting at vertex a and ending at b) in G.

HAMILTON-CYCLE vs. TSP(D)

HAMILTON-CYCLE $\leq_{\mathrm{L}}$ TSP(D)

Let an arbitrary instance of **HAMILTON-CYCLE** be given by the graph $G = (V, E)$. We construct an equivalent instance of **TSP(D)** as follows:

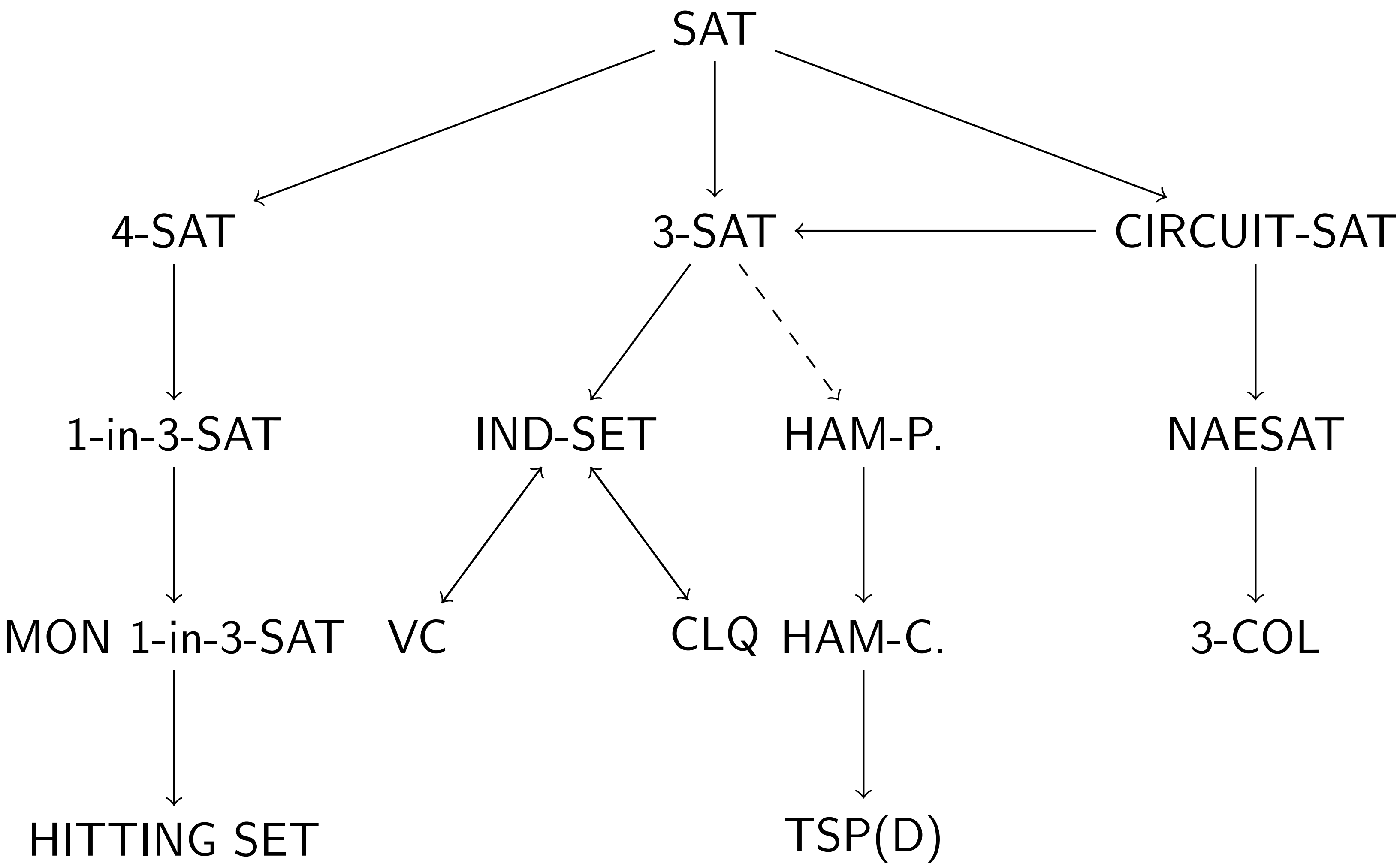
Let $V = \{1, \dots, n\}$. Then our instance of **TSP(D)** has n cities. Moreover, for any two cities $i \neq j$, the distance is defined as

$$d_{ij} = \begin{cases} 1 & \text{if } [i,j] \in E \\ 2 & \text{otherwise} \end{cases}$$

Finally, we set $B = n$.

Clearly, there is no tour through all cities of length $< B = n$.
Moreover, the Hamilton cycles in G are precisely the tours of length B.
Hence, G has a Hamilton cycle $\Leftrightarrow$ there exists a tour of length $\leq B$.

Summary of Reductions



Learning Objectives

- The concept of NP-completeness and its characterizations in terms of succinct certificates.
- You should now be familiar with the intuition of NP-completeness (and recognize NP-complete problems)
- Basic techniques to prove problems NP-complete
- A basic repertoire of NP-complete problems (in particular, versions of **SAT** and some graph problems) to be used in further NP-completeness proofs.
- Reductions, reductions, reductions, . . .