Complexity Theory

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3. Logarithmic Space

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Outline

Example

Log-space computations

If a Turing machine with input and output is supposed to operate within space bound $\mathrm{O}(\log(n))$, then it may never copy a substantial portion of the input onto its worktapes. Access to the input and control of the machine is based on the following log-space functionalities:

- maintaining (a constant number of) counters
- maintaining (a constant number of) "pointers" to the input
- "storing" a constant-size fragment of the input in the state

Example

The language of palindromes (over an arbitrary alphabet Σ) can be recognized by a 3-string Turing machine M with input within space bound $\mathrm{O}(\log(n))$. (There is no output tape needed.)

Sketch of a log-space TM for palindromes

The 3-string Turing machine M with input implements a loop from 1 to n with $n = |x|$. For every i, it checks if $x[i] = x[n+1-i]$ holds:

1. Tape 2 contains the loop counter i. It is initialized to 1 (in binary) and will be subsequently incremented.
2. Tape 3 contains another counter j which, for every i, controls the cursor movements required for comparing $x[i]$ and $x[n+1-i]$:
   (a) Initialize j to 1.
   (b) Move the cursor from the front end of x to $x[i]$. The integer j counts the cursor movements (i.e.: increment j as long as $j < i$).
   (c) "Store" the symbol $x[i]$ in an appropriate state.
   (d) Reinitialize j to 1.
   (e) Move the cursor from the rear end of x to $x[n+1-i]$.
   (f) "Compare" $x[n+1-i]$ with $x[i]$ (via the state). If $x[n+1-i] = x[i]$ then increment i and repeat Step 2. Otherwise halt with "no".

Log-Space Reductions

Reductions in the "Formale Methoden" lecture

- 2 kinds of reductions: Turing reductions vs. many-one reductions
- Limit on the resources needed by a reduction: polynomial time vs. logarithmic space reductions.
- Default for problem reductions in "Formale Methoden" (e.g., in NP-completeness proofs) : polynomial time, many-one reductions.
- In this course: We also want to prove completeness results for classes below NP (in particular, completeness in P or NL).
  $\Longrightarrow$ We need reductions in a complexity class below P.
- From now on: log-space, many-one reductions, denoted as $\leq_{\mathrm{L}}$
- Remark. All polynomial-time reductions encountered so far (in the "Formale Methoden" lecture) also work in log-space!

Composing Reductions

- In the "Formale Methoden" lecture, we have established the following chain of reductions:
  3-SAT $\leq_{\mathrm{L}}$ **INDEPENDENT SET** $\leq_{\mathrm{L}}$ **VERTEX COVER**.
- But do reductions compose, i.e., is $\leq_{\mathrm{L}}$ transitive?
  For instance, does **3-SAT** $\leq_{\mathrm{L}}$ **VERTEX COVER** hold?

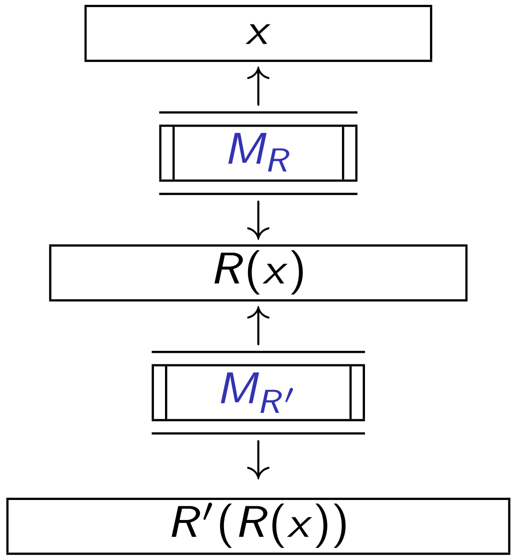
Theorem

If R is a reduction from language L_1 to L_2 and R' is a reduction from L_2 to L_3, then the composition $R \cdot R'$ is a reduction from L_1 to L_3.

- As R, R' are reductions, $x \in L_1$ iff $R(x) \in L_2$ iff $R'(R(x)) \in L_3$.
- It remains to show that $R'(R(x))$ can be computed in $\mathrm{O}(\log n)$ space where $n = |x|$.

Logarithmic space consumption

- To construct a machine M for the composition $R \cdot R'$ working in space $\mathrm{O}(\log n)$ requires care as the intermediate result computed by M_R cannot be stored (possibly longer than $\log n$).
- Solution: simulate $M_{R'}$ on input $R(x)$ by remembering the cursor position i of the input string of $M_{R'}$ which is the output string of M_R. Only the index i is stored (in binary) and the symbol currently scanned but not the whole string.



Logarithmic space consumption (continued)

- Initially $i = 1$ and it is easy to simulate the first move of $M_{R'}$ (scanning $\triangleright$).
- If $M_{R'}$ moves the cursor on the input tape to the right, then simulate M_R to generate the next output symbol and increment i by one.
- If $M_{R'}$ moves the cursor on the input tape to the left, then decrement i by one and run M_R on x from the beginning, counting the output symbols and stopping when the i-th symbol is output.
- The space required for simulating M_R on x as well as $M_{R'}$ on $R(x)$ is $\mathrm{O}(\log n)$ where $n = |x|$.
- The space needed for bookkeeping the output of M_R on x is $\mathrm{O}(\log n)$ as $|R(x)| = \mathrm{O}(n^k)$ and we need only indices stored in binary.

Nondeterministic Log-Space

Motivation

- The intuition of a complexity class is best understood by looking at "natural" problems which are complete for this class.
- The complexity of a problem is only understood if we manage to show its completeness in a "natural" complexity class.

Theorem

REACHABILITY *is* NL*-complete (w.r.t. log-space reductions), i.e.*

- *It can be decided by an NTM in space* $\mathrm{O}(\log(n))$*.*
- *Any problem in* NL *can be reduced to it in log-space.*

Proof sketch of the NL-membership

Let (V, E) be a graph with vertices $V = \{1, \ldots, n\}$. Moreover, suppose that we attempt to find a path from vertex 1 to n. We sketch an NTM N with input tape plus 3 worktapes:

1. On tape 2, store the current node i in binary; initially take node 1.
2. On tape 3, "guess" an integer $j \leq n$ (= next node in the path) and check that $(i, j) \in E$.
3. If $(i, j) \in E$, then continue at step 2 with j as the new current node. Moreover, if $j = n$, then halt with "yes".
   If $(i, j) \notin E$, then halt with "no".
4. On tape 4, maintain a counter which checks that we do not construct paths of length $\geq n$.

Proof sketch of the NL-hardness

Let $\mathcal{P}$ be an arbitrary problem in NL, i.e., $\mathcal{P}$ is decided by a k-string *nondeterministic* TM M with input tape within space $f(n) = \mathrm{O}(\log n)$.

Let x be an arbitrary instance of $\mathcal{P}$. From this, we construct the instance $R(x) = (G, u, v)$ of **REACHABILITY** as follows:

The configuration graph $G(M, x)$ of M has as its nodes all possible configurations of M and there is an edge between two nodes (configurations) C_1 and C_2 iff $C_1 \stackrel{M}{\rightarrow} C_2$.

Our graph G contains an additional node "success" and there is an edge from any configuration C with state "yes" to the "success" node.

Finally, we set $u = C_0 = (s, \triangleright, x, \triangleright, \epsilon, \ldots, \triangleright, \epsilon)$ (= the initial configuration of M on input x) and $v =$ "success".

Proof sketch of the NL-hardness (continued)

Clearly, this problem reduction is correct, i.e., there exists an accepting computation of M on input x iff there exists a path in graph G from node $u = C_0$ to node $v =$ "success".

It remains to show that $R(x)$ can be computed (by a deterministic TM N with input and output) in log-space.

Sketch of a log-space TM N.

- N has 3 worktapes. The first two worktapes are used to store one possible configuration of M at a time.
- We are assuming that M is a k-string Turing machine with input. Hence, configurations are represented as $(q, i, w_2, u_2, \dots, w_k, u_k)$ where $1 \leq i \leq n + 1$ gives the cursor position on the input string. Clearly, position i as well as each string u_j, w_j fits into log-space.

Proof sketch of the NL-hardness (continued)

Operating principles of the log-space TM N.

- In a loop, N generates all possible configurations of M on its first worktape and writes each configuration to the output tape. At the end, also "success" is output.
- In a nested loop, N generates all possible configurations of M plus the additional node "success" on each of its first two worktapes.
- For every pair of configurations C_1 and C_2, N checks (by using the third worktape) if $C_1 \xrightarrow{M} C_2$ holds. If this is the case, then N writes the pair (= edge) (C_1, C_2) to the output.
- N also outputs all edges $(C,$ "success"$)$ if C corresponds to an accepting configuration.
- Finally, the nodes $u = C_0 = (s, 0, \triangleright, \epsilon, \ldots, \triangleright, \epsilon)$ and $v =$ "success" are output.

Nondeterministic Space vs. Deterministic Time

Motivation

- The NL-hardness proof of **REACHABILITY** implicitly establishes that NL $\subseteq$ P holds.
- This result can be rephrased as follows: a nondeterministic machine M working in space $f(n) = \log n$ can be simulated by a deterministic machine N, s.t. the time bound of N is exponential w.r.t. $f(n)$.
- We want to generalize this relationship between space and time complexity to any "reasonable" function $f(n) \geq \log n$.
- The idea of "reasonable" functions is formalized by the definition of proper complexity functions.

Proper Complexity Functions

Definition

A function $f : \mathbf{N} \to \mathbf{N}$ is a proper complexity function if f is nondecreasing and there is a k-string TM M_f with input and output such that on any input x,

1. $M_f(x) = \sqcap^{f(|x|)}$ where $\sqcap$ is a *quasi-blank* symbol,
2. M_f halts after $\mathrm{O}(|x| + f(|x|))$ steps, and
3. M_f uses $\mathrm{O}(f(|x|))$ space besides its input.

Remark

For the definition of complexity classes $\mathrm{TIME}(f(n))$, $\mathrm{NTIME}(f(n))$, $\mathrm{SPACE}(f(n))$, $\mathrm{NSPACE}(f(n))$, we have to avoid the situation that a function f cannot be computed within the time or space it allows.

Theorem

For any proper complexity function $f(n) \geq \log n$, we have

$$\text{NSPACE}(f(n)) \subseteq \text{TIME}(c^{f(n)})$$

Proof sketch

- We construct the configuration graph as in the NL-hardness proof of **REACHABILITY**.
- We get the following bound on the number of possible configurations represented as $(q, i, w_2, u_2, \ldots, w_k, u_k)$ where $1 \leq i \leq n+1$:

  $|K|(n+1)(|\Sigma|^{f(n)})^{2(k-1)} \leq |K|2n(|\Sigma|^{2(k-1)})^{f(n)} \leq$

  $nc_1^{f(n)} \leq c_1^{\log n + f(n)} \leq c_1^{2f(n)} \leq c^{f(n)}$
- Hence, deciding if x is a positive instance of problem $\mathcal{P}$ reduces to a reachability problem for a graph with at most $c^{f(n)}$ nodes.

Savitch's Theorem

Theorem

REACHABILITY $\in$ SPACE($\log^2 n$).

Proof sketch

- Given a graph G and nodes x, y and $i \geq 0$, define $PATH(x, y, i)$ as the assertion "there is a path from x to y of length at most 2^i".
- If G has n nodes then any cycle-free path has length $\leq n$ and we can solve reachability in G if we can compute whether $PATH(x, y, \lceil \log n \rceil)$ holds for any given nodes x, y of G.
- This can be done using middle-first search.

Proof sketch (continued)

- Idea of the middle-first search. Guess a midpoint of the alleged path and recursively check for the existence of a half-length path from the start to the midpoint and from the midpoint to the finish.
- Implementation of the middle-first search.

function $path(x, y, i)$ /* middle-first search */
if $i = 0$ **then**
  if $x = y$ or there is an edge (x, y) in G **then** return "yes";
else for all nodes z **do**
    if $path(x, z, i-1)$ and $path(z, y, i-1)$ **then** return "yes";
return "no".

Proof sketch (continued)

- Proof that $path(x, y, i)$ correctly determines $PATH(x, y, i)$ (by induction on i):

  $i = 0$. Clearly, $path(x, y, 0)$ correctly determines $PATH(x, y, 0)$.

  $i > 0$. $path(x, y, i)$ returns "yes" iff there is a node z with both $path(x, z, i-1)$ and $path(z, y, i-1)$ holding.

  By the induction hypothesis, $path(x, z, i-1)$ and $path(z, y, i-1)$ return "yes" iff there are paths from x to z and from z to y both at most 2^{i-1} long.

  This is the case iff there is a path from x to y at most 2^i long.

Proof sketch (continued)

- The algorithm is started with $\mathit{path}(x, y, \lceil \log n \rceil)$.
- The $\mathrm{O}(\log^2 n)$ space bound is achieved by handling recursion using a stack containing a triple (x, y, i) for each active recursive call.

  For each node z put $(x, z, i-1)$ onto the stack and call $\mathit{path}(x, z, i-1)$.

  If this fails, erase $(x, z, i-1)$ and put $(x, z', i-1)$ for the next z'. Otherwise erase $(x, z, i-1)$ and continue with $(z, y, i-1)$.
- As there are at most $\log n$ recursive calls active with each taking at most $3 \log n$ space, the $\mathrm{O}(\log^2 n)$ space bound is achieved.

Nondeterministic Space vs. Deterministic Space

Corollary

For any proper complexity function $f(n) \geq \log n$, *we have*

$$\text{NSPACE}(f(n)) \subseteq \text{SPACE}((f(n))^2)$$

Proof sketch

- To simulate an $f(n)$-space bounded NTM M on input x, run the previous algorithm on the configuration graph $G(M, x)$.
- The edges of the graph $G(M, x)$ are determined on the fly by examining the input x.
- The configuration graph has at most $c^{f(n)}$ nodes.
- By Savitch's Theorem, the algorithm needs at most $(\log c^{f(n)})^2 = f(n)^2 \log^2 c = \mathrm{O}(f(n)^2)$ space.

Basic Complexity Classes Revisited

Theorem

$\mathrm{L} \subseteq \mathrm{NL} \subseteq \mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE} \subseteq \mathrm{EXPTIME} \subseteq \mathrm{NEXPTIME} \subseteq \mathrm{EXPSPACE}$

Proof

In the "Formale Methoden" lecture, we showed all inclusions except for $\mathrm{NL} \subseteq \mathrm{P}$ and $\mathrm{PSPACE} \subseteq \mathrm{EXPTIME}$.

These inclusions follow from the more general relationship

$$\mathrm{NSPACE}(f(n)) \subseteq \mathrm{TIME}(c^{f(n)})$$

Remark

It is now clear, why we have not mentioned NPSPACE and NEXPSPACE. Indeed, $\mathrm{PSPACE} = \mathrm{NPSPACE}$ and $\mathrm{EXPSPACE} = \mathrm{NEXPSPACE}$ hold.

Immerman-Szelepscényi Theorem

Theorem

Given a graph G and a node u, the number of nodes reachable from u in G can be computed by an NTM within logarithmic space.

More formally, given a graph G, a node u, and an integer m, deciding if the number of nodes reachable from u is m can be done in NL.

Theorem

REACHABILITY *is in* co-NL. *Hence,* NL = co-NL.

Theorem

If $f(n) \geq \log n$ is a proper complexity function, then $\mathrm{NSPACE}(f(n)) = \text{co-}\mathrm{NSPACE}(f(n))$.

Witnesses for NL-Problems

Characterization of positive instances

Let $\mathcal{P}$ denote a problem in NL.
Every positive instance x of $\mathcal{P}$ can be characterized by a witness y, s.t.

- the witness y may be polynomially big,
- but we can check if y is a witness by a sequence of local consistency checks – each requiring only logarithmic space.

REACHABILITY

(G, u, v) is a positive instance of **REACHABILITY** if there exists a path π from u to v s.t.

- the path $\pi = (z_0, z_1, \ldots, z_{k-1}, z_k)$ with $u = z_0$, $v = z_k$, and $k \leq n$ with $n = |V|$ may be polynomially big,
- but it suffices to check for every pair of neighbouring nodes z_i, z_{i+1} that an arc (z_i, z_{i+1}) indeed exists in E.

co-NL-Membership of REACHABILITY

Proof Idea

What shall a witness look like that there is **no** path from u to v?

Idea. Let $S(k)$ denote the set of nodes reachable from u in k steps (with $0 \leq k \leq n-1$) and suppose that we know $m = |S(n-1)|$. Then a witness y for not-reachability consists of m paths from u to (pairwise distinct) vertices $v_1, \ldots, v_m \in V$, s.t. $v_i \neq v$ for all i.

Logspace verification of such witnesses

for $j := 1, 2, \ldots, n$ **do** {
  guess flag; /* meaning: flag = true if $v_j \in S(n-1)$ */
  if flag **then** {
    decrement m;
    guess a path π from u to v_j of length $\leq n-1$;
    check that $v \neq v_j$; } }
check that $m = 0$;

Counting the Number of Reachable Nodes in "NL"

Idea of the Algorithm

The strategy is to compute values $|S(1)|, |S(2)|, \ldots, |S(n-1)|$ iteratively and recursively, i.e. $|S(i)|$ is computed from $|S(i-1)|$.

$|S(0)| := 1$;
for $k := 1, 2, ..., n-1$ **do** {
  $\ell := 0$;
  for $j := 1, 2, ..., n$ **do**
    guess flag;   /* meaning: flag = true if $v_j \in S(k)$ */
    if flag **then** { check that $v_j \in S(k)$; increment ℓ; }
    else check that $v_j \notin S(k)$;
  }
  $|S(k)| := \ell$;
}

Idea of the Algorithm (continued)

- Clearly, we can check $v_j \in S(k)$ in NL. It remains to show that we can also check $v_j \notin S(k)$ in NL in the inner loop of our algorithm.
- When checking $v_j \notin S(k)$, we already know $m = |S(k-1)|$.
- A witness y for $v_j \notin S(k)$ consists of m paths from u to vertices $v_1, \dots, v_m \in V$, s.t. $v_i \neq v_j$ and $(v_i, v_j) \notin E$.
- Below we give a verification of witness y that requires only logspace.

```
for v := 1, 2, ..., n do {
   guess flag;    /* meaning: flag = true if v ∈ S(k − 1) */
   if flag then {
      decrement m;
      guess a path π from u to v of length ≤ k − 1;
      check that v ≠ v_j and (v, v_j) ∉ E; } }
check that m = 0;
```

Learning Objectives

- Computational power of log-space
- Reductions in this lecture: log-space, many-one
- Composability of reductions
- NL-completeness of **REACHABILITY** (configuration graph)
- Savitch's Theorem (middle-first search)
- Basic relationships (inclusions) between complexity classes
- Immerman-Szelepscényi Theorem