# Database Theory

## VU 181.140, WS 2024

## 7. Ehrenfeucht-Fraïssé Games

Matthias Lanzinger
(adapted from slides by Reinhard Pichler/Christoph Koch)

Institut für Informationssysteme
Arbeitsbereich DBAI
Technische Universität Wien

10 December, 2024

TU WIEN Informatics

# Motivation

# Using logic to express properties of structures

## Definition

Let $\mathcal{L}$ be some logic. We say that some property $\mathcal{P}$ of structures is expressible in $\mathcal{L}$ if there exists a sentence $\phi$ in $\mathcal{L}$, s.t. for all structures $\mathcal{A}$, the following equivalence holds:

$$\mathcal{A} \text{ has property } \mathcal{P} \text{ iff } \mathcal{A} \models \phi$$

# Using logic to express properties of structures

## Definition

Let $\mathcal{L}$ be some logic. We say that some property $\mathcal{P}$ of structures is expressible in $\mathcal{L}$ if there exists a sentence $\phi$ in $\mathcal{L}$, s.t. for all structures $\mathcal{A}$, the following equivalence holds:

$$\mathcal{A} \text{ has property } \mathcal{P} \text{ iff } \mathcal{A} \models \phi$$

## Example

Property: "graph is closed w.r.t. transitivity":

This property is expressible in First-Order logic:

# Using logic to express properties of structures

## Definition

Let $\mathcal{L}$ be some logic. We say that some property $\mathcal{P}$ of structures is expressible in $\mathcal{L}$ if there exists a sentence $\phi$ in $\mathcal{L}$, s.t. for all structures $\mathcal{A}$, the following equivalence holds:

$$\mathcal{A} \text{ has property } \mathcal{P} \text{ iff } \mathcal{A} \models \phi$$

## Example

Property: "graph is closed w.r.t. transitivity":

This property is expressible in First-Order logic:

$$\phi = \forall x \forall y \forall z \big(e(x, y) \land e(y, z) \to e(x, z)\big)$$

# Motivation

- Goal: Inexpressibility proofs for FO queries.
- A standard technique for inexpressibility proofs from logic (model theory): Compactness theorem.
  - Discussed in logic lectures.
  - Fails if we are only interested in finite structures (=databases).
    The compactness theorem does not hold in the finite!
- We need a different technique to prove that certain queries are not expressible in FO.
- EF games are such a technique.
- We will then also have a glimpse beyond FO to MSO.

# Inexpressibility via Compactness Theorem

## Theorem (Compactness)

*Let $\Phi$ be an infinite set of FO sentences and suppose that every finite subset of $\Phi$ is satisfiable. Then also $\Phi$ is satisfiable.*

## Definition

Property CONNECTED: Does there exists a (finite) path between any two nodes $u, v$ in a given (possibly infinite) graph?

## Theorem

*CONNECTED is not expressible in FO, i.e., there does not exist an FO sentence $\psi$, s.t. for every structure $\mathcal{G}$ representing a graph, the following equivalence holds:*

$$\text{Graph } \mathcal{G} \text{ is connected iff } \mathcal{G} \models \psi.$$

## Proof

Assume to the contrary that there exists an FO-formula $\psi$ which expresses CONNECTED. We derive a contradiction as follows.

1. Extend the vocabulary of graphs by two constants $c_1$ and $c_2$ and consider the set of formulae $\Phi = \{\psi\} \cup \{\phi_n \mid n \geq 1\}$ with

$$\phi_n := \neg \exists x_1 \ldots \exists x_n\ x_1 = c_1 \land x_n = c_2 \land \bigwedge_{1 \leq i \leq n-1} E(x_i, x_{i+1}).$$

("There does not exist a path of length $n - 1$ between $c_1$ and $c_2$".)

## Proof

Assume to the contrary that there exists an FO-formula $\psi$ which expresses CONNECTED. We derive a contradiction as follows.

1. Extend the vocabulary of graphs by two constants $c_1$ and $c_2$ and consider the set of formulae $\Phi = \{\psi\} \cup \{\phi_n \mid n \geq 1\}$ with

$$\phi_n := \neg\exists x_1 \ldots \exists x_n \; x_1 = c_1 \wedge x_n = c_2 \wedge \bigwedge_{1 \leq i \leq n-1} E(x_i, x_{i+1}).$$

   ("There does not exist a path of length $n-1$ between $c_1$ and $c_2$".)

2. Clearly, $\Phi$ is unsatisfiable.

## Proof

Assume to the contrary that there exists an FO-formula $\psi$ which expresses CONNECTED. We derive a contradiction as follows.

1. Extend the vocabulary of graphs by two constants $c_1$ and $c_2$ and consider the set of formulae $\Phi = \{\psi\} \cup \{\phi_n \mid n \geq 1\}$ with

$$\phi_n := \neg \exists x_1 \ldots \exists x_n \, x_1 = c_1 \wedge x_n = c_2 \wedge \bigwedge_{1 \leq i \leq n-1} E(x_i, x_{i+1}).$$

   ("There does not exist a path of length $n-1$ between $c_1$ and $c_2$".)

2. Clearly, $\Phi$ is unsatisfiable.
3. Consider an arbitrary, finite subset $\Phi_0$ of $\Phi$. There exists $n_{\max}$, s.t. $\phi_m \notin \Phi_0$ for all $m > n_{\max}$.

## Proof

Assume to the contrary that there exists an FO-formula $\psi$ which expresses CONNECTED. We derive a contradiction as follows.

1. Extend the vocabulary of graphs by two constants $c_1$ and $c_2$ and consider the set of formulae $\Phi = \{\psi\} \cup \{\phi_n \mid n \geq 1\}$ with

$$\phi_n := \neg \exists x_1 \ldots \exists x_n\, x_1 = c_1 \wedge x_n = c_2 \wedge \bigwedge_{1 \leq i \leq n-1} E(x_i, x_{i+1}).$$

   ("There does not exist a path of length $n-1$ between $c_1$ and $c_2$".)

2. Clearly, $\Phi$ is unsatisfiable.
3. Consider an arbitrary, finite subset $\Phi_0$ of $\Phi$. There exists $n_{\max}$, s.t. $\phi_m \notin \Phi_0$ for all $m > n_{\max}$.
4. $\Phi_0$ is satisfiable: indeed, a single path of length $n_{\max} + 1$ (where we interpret $c_1$ and $c_2$ as the endpoints of this path) satisfies $\Phi_0$.

## Proof

Assume to the contrary that there exists an FO-formula $\psi$ which expresses CONNECTED. We derive a contradiction as follows.

1. Extend the vocabulary of graphs by two constants $c_1$ and $c_2$ and consider the set of formulae $\Phi = \{\psi\} \cup \{\phi_n \mid n \geq 1\}$ with

$$\phi_n := \neg\exists x_1 \ldots \exists x_n\ x_1 = c_1 \land x_n = c_2 \land \bigwedge_{1 \leq i \leq n-1} E(x_i, x_{i+1}).$$

   ("There does not exist a path of length $n-1$ between $c_1$ and $c_2$".)
2. Clearly, $\Phi$ is unsatisfiable.
3. Consider an arbitrary, finite subset $\Phi_0$ of $\Phi$. There exists $n_{\max}$, s.t. $\phi_m \notin \Phi_0$ for all $m > n_{\max}$.
4. $\Phi_0$ is satisfiable: indeed, a single path of length $n_{\max} + 1$ (where we interpret $c_1$ and $c_2$ as the endpoints of this path) satisfies $\Phi_0$.
5. By the Compactness Theorem, $\Phi$ is satisfiable, which contradicts the observation (2) above. Hence, $\psi$ cannot exist. □

# Compactness over Finite Models

Question.
Does the theorem also establish that connectedness of finite graphs is FO inexpressible?
The answer is “no”!

# Compactness over Finite Models

**Question.**
Does the theorem also establish that connectedness of finite graphs is FO inexpressible?
The answer is "no"!

## Proposition

Compactness fails over finite models, i.e., there exists a set $\Phi$ of FO sentences with the following properties:

- every finite subset of $\Phi$ has a finite model and
- $\Phi$ has no finite model.

# Compactness over Finite Models

## Proof

Consider the set $\Phi = \{d_n \mid n \geq 2\}$ with $d_n := \exists x_1 \ldots \exists x_n \bigwedge_{i \neq j} x_i \neq x_j$,
i.e., $d_n \Leftrightarrow$ there exist at least $n$ pairwise distinct elements.

Clearly, every finite subset $\Phi_0 = \{d_{i_1}, \ldots, d_{i_k}\}$ of $\Phi$ has a finite model: just take a set whose cardinality exceeds $\max(\{i_1, \ldots, i_k\})$.
However, $\Phi$ does not have a finite model. □

# Rules of the EF game

# Rules of the EF game

- Two players: Spoiler S, Duplicator D.
- "Game board": Two structures of the same schema.
- Players move alternatingly; Spoiler starts (like in chess).
- The number of moves $k$ to be played is fixed in advance (differently from chess).
- Tokens $S_1, \ldots, S_k, D_1, \ldots, D_k$.

# Rules of the EF game

- Two players: Spoiler S, Duplicator D.
- "Game board": Two structures of the same schema.
- Players move alternatingly; Spoiler starts (like in chess).
- The number of moves $k$ to be played is fixed in advance (differently from chess).
- Tokens $S_1, \ldots, S_k, D_1, \ldots, D_k$.
- In the $i$-th move, Spoiler first selects a structure and places token $S_i$ on a domain element of that structure. Next, Duplicator places token $D_i$ on an arbitrary domain element of the other structure. (That's one move, not two.)

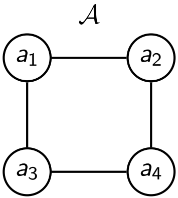
# Rules of the EF game

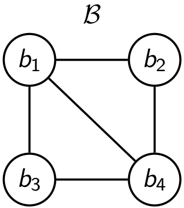
- Two players: Spoiler S, Duplicator D.
- "Game board": Two structures of the same schema.
- Players move alternatingly; Spoiler starts (like in chess).
- The number of moves $k$ to be played is fixed in advance (differently from chess).
- Tokens $S_1, \ldots, S_k, D_1, \ldots, D_k$.
- In the $i$-th move, Spoiler first selects a structure and places token $S_i$ on a domain element of that structure. Next, Duplicator places token $D_i$ on an arbitrary domain element of the other structure. (That's one move, not two.)
- Spoiler may choose its structure anew in each move. Duplicator always has to answer in the other structure.
- A token, once placed, cannot be (re)moved.
- The winning condition follows a bit later.

# Notation from Finite Model Theory

- $\mathcal{A}, \mathcal{B}$ denote structures (= databases),
- $|\mathcal{A}|$ is the domain of a structure $\mathcal{A}$,
- $E^{\mathcal{A}}$ is the relation $E$ of a structure $\mathcal{A}$.

# A game run with $k = 3$



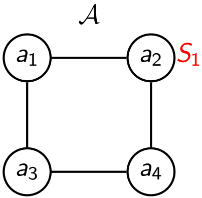

| $E^{\mathcal{A}}$ | | |
|---|---|---|
| | $a_1$ | $a_2$ |
| | $a_2$ | $a_1$ |
| | $\vdots$ | $\vdots$ |
| | $a_4$ | $a_3$ |

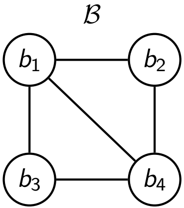
| $\lvert\mathcal{A}\rvert$ | |
|---|---|
| | $a_1$ |
| | $a_2$ |
| | $a_3$ |
| | $a_4$ |

| $E^{\mathcal{B}}$ | | |
|---|---|---|
| | $b_1$ | $b_2$ |
| | $b_2$ | $b_1$ |
| | $\vdots$ | $\vdots$ |
| | $b_4$ | $b_3$ |
| | $b_1$ | $b_4$ |
| | $b_4$ | $b_1$ |

| $\lvert\mathcal{B}\rvert$ | |
|---|---|
| | $b_1$ |
| | $b_2$ |
| | $b_3$ |
| | $b_4$ |

# A game run with $k = 3$



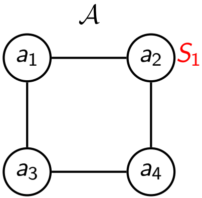

| $E^{\mathcal{A}}$ | | |
|---|---|---|
| | $a_1$ | $a_2$ |
| | $a_2$ | $a_1$ |
| | $\vdots$ | $\vdots$ |
| | $a_4$ | $a_3$ |

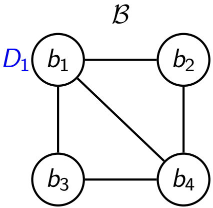
| $\lvert\mathcal{A}\rvert$ | |
|---|---|
| | $a_1$ |
| $S_1$ | $a_2$ |
| | $a_3$ |
| | $a_4$ |

| $E^{\mathcal{B}}$ | | |
|---|---|---|
| | $b_1$ | $b_2$ |
| | $b_2$ | $b_1$ |
| | $\vdots$ | $\vdots$ |
| | $b_4$ | $b_3$ |
| | $b_1$ | $b_4$ |
| | $b_4$ | $b_1$ |

| $\lvert\mathcal{B}\rvert$ | |
|---|---|
| | $b_1$ |
| | $b_2$ |
| | $b_3$ |
| | $b_4$ |

# A game run with $k = 3$



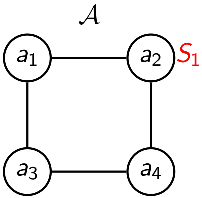

| $E^{\mathcal{A}}$ | | |
|---|---|---|
| | $a_1$ | $a_2$ |
| | $a_2$ | $a_1$ |
| | $\vdots$ | $\vdots$ |
| | $a_4$ | $a_3$ |

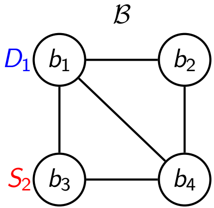
| $\lvert\mathcal{A}\rvert$ | |
|---|---|
| | $a_1$ |
| $S_1$ | $a_2$ |
| | $a_3$ |
| | $a_4$ |

| $E^{\mathcal{B}}$ | | |
|---|---|---|
| | $b_1$ | $b_2$ |
| | $b_2$ | $b_1$ |
| | $\vdots$ | $\vdots$ |
| | $b_4$ | $b_3$ |
| | $b_1$ | $b_4$ |
| | $b_4$ | $b_1$ |

| $\lvert\mathcal{B}\rvert$ | |
|---|---|
| $D_1$ | $b_1$ |
| | $b_2$ |
| | $b_3$ |
| | $b_4$ |

# A game run with $k = 3$



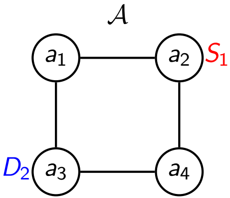

| $E^{\mathcal{A}}$ | | |
|---|---|---|
| | $a_1$ | $a_2$ |
| | $a_2$ | $a_1$ |
| | $\vdots$ | $\vdots$ |
| | $a_4$ | $a_3$ |

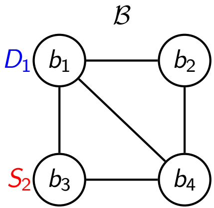
| $\lvert\mathcal{A}\rvert$ | |
|---|---|
| | $a_1$ |
| $S_1$ | $a_2$ |
| | $a_3$ |
| | $a_4$ |

| $E^{\mathcal{B}}$ | | |
|---|---|---|
| | $b_1$ | $b_2$ |
| | $b_2$ | $b_1$ |
| | $\vdots$ | $\vdots$ |
| | $b_4$ | $b_3$ |
| | $b_1$ | $b_4$ |
| | $b_4$ | $b_1$ |

| $\lvert\mathcal{B}\rvert$ | |
|---|---|
| $D_1$ | $b_1$ |
| | $b_2$ |
| $S_2$ | $b_3$ |
| | $b_4$ |

# A game run with $k = 3$



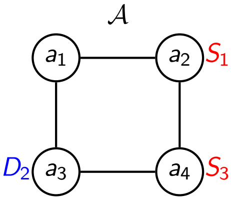

| $E^{\mathcal{A}}$ | | |
|---|---|---|
| | $a_1$ | $a_2$ |
| | $a_2$ | $a_1$ |
| | $\vdots$ | $\vdots$ |
| | $a_4$ | $a_3$ |

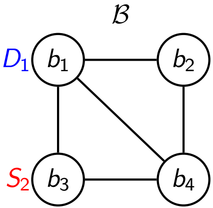
| $\lvert\mathcal{A}\rvert$ | |
|---|---|
| | $a_1$ |
| $S_1$ | $a_2$ |
| $D_2$ | $a_3$ |
| | $a_4$ |

| $E^{\mathcal{B}}$ | | |
|---|---|---|
| | $b_1$ | $b_2$ |
| | $b_2$ | $b_1$ |
| | $\vdots$ | $\vdots$ |
| | $b_4$ | $b_3$ |
| | $b_1$ | $b_4$ |
| | $b_4$ | $b_1$ |

| $\lvert\mathcal{B}\rvert$ | |
|---|---|
| $D_1$ | $b_1$ |
| | $b_2$ |
| $S_2$ | $b_3$ |
| | $b_4$ |

# A game run with $k = 3$



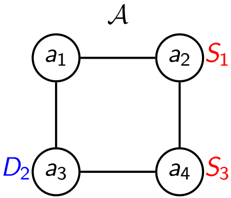

| $E^{\mathcal{A}}$ | | |
|---|---|---|
| | $a_1$ | $a_2$ |
| | $a_2$ | $a_1$ |
| | $\vdots$ | $\vdots$ |
| | $a_4$ | $a_3$ |

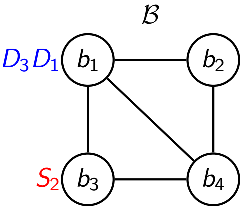
| $\lvert\mathcal{A}\rvert$ | |
|---|---|
| | $a_1$ |
| $S_1$ | $a_2$ |
| $D_2$ | $a_3$ |
| $S_3$ | $a_4$ |

| $E^{\mathcal{B}}$ | | |
|---|---|---|
| | $b_1$ | $b_2$ |
| | $b_2$ | $b_1$ |
| | $\vdots$ | $\vdots$ |
| | $b_4$ | $b_3$ |
| | $b_1$ | $b_4$ |
| | $b_4$ | $b_1$ |

| $\lvert\mathcal{B}\rvert$ | |
|---|---|
| $D_1$ | $b_1$ |
| | $b_2$ |
| $S_2$ | $b_3$ |
| | $b_4$ |

# A game run with $k = 3$




| $E^{\mathcal{A}}$ | | |
|---|---|---|
| | $a_1$ | $a_2$ |
| | $a_2$ | $a_1$ |
| | ⋮ | ⋮ |
| | $a_4$ | $a_3$ |

| $\lvert\mathcal{A}\rvert$ | |
|---|---|
| | $a_1$ |
| $S_1$ | $a_2$ |
| $D_2$ | $a_3$ |
| $S_3$ | $a_4$ |

| $E^{\mathcal{B}}$ | | |
|---|---|---|
| | $b_1$ | $b_2$ |
| | $b_2$ | $b_1$ |
| | ⋮ | ⋮ |
| | $b_4$ | $b_3$ |
| | $b_1$ | $b_4$ |
| | $b_4$ | $b_1$ |

| $\lvert\mathcal{B}\rvert$ | |
|---|---|
| $D_3\, D_1$ | $b_1$ |
| | $b_2$ |
| $S_2$ | $b_3$ |
| | $b_4$ |

# Partial isomorphisms

## Definition

- $\mathcal{A}|_S$: Restriction of a structure $\mathcal{A}$ to the subdomain $S \subseteq |\mathcal{A}|$. Same schema; for each relation $R^{\mathcal{A}}$:

$$R^{\mathcal{A}|_S} := \{\langle a_1, \ldots, a_k \rangle \in R^{\mathcal{A}} \mid a_1, \ldots, a_k \in S\}.$$

- A partial function $\theta : |\mathcal{A}| \to |\mathcal{B}|$ is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ if and only if $\theta$ is an isomorphism from $\mathcal{A}|_{\mathrm{dom}(\theta)}$ to $\mathcal{B}|_{\mathrm{rng}(\theta)}$.
- This definition assumes that the schema of $\mathcal{A}$ does not contain any constants but is purely relational.

# Partial isomorphisms

## Example

| $R^{\mathcal{A}}$ | | | |
|---|---|---|---|
| | 1 | 2 | 3 |
| | 2 | 1 | 4 |

| $\lvert\mathcal{A}\rvert$ | |
|---|---|
| | 1 |
| | 2 |
| | 3 |
| | 4 |

| $R^{\mathcal{B}}$ | | | |
|---|---|---|---|
| | $a$ | $b$ | $c$ |
| | $a$ | $b$ | $d$ |

| $\lvert\mathcal{B}\rvert$ | |
|---|---|
| | $a$ |
| | $b$ |
| | $c$ |
| | $d$ |

# Partial isomorphisms

## Example

| $R^{\mathcal{A}}$ | | |
|---|---|---|
| 1 | 2 | 3 |
| 2 | 1 | 4 |

| $\lvert\mathcal{A}\rvert$ |
|---|
| 1 |
| 2 |
| 3 |
| 4 |

| $R^{\mathcal{B}}$ | | |
|---|---|---|
| $a$ | $b$ | $c$ |
| $a$ | $b$ | $d$ |

| $\lvert\mathcal{B}\rvert$ |
|---|
| $a$ |
| $b$ |
| $c$ |
| $d$ |

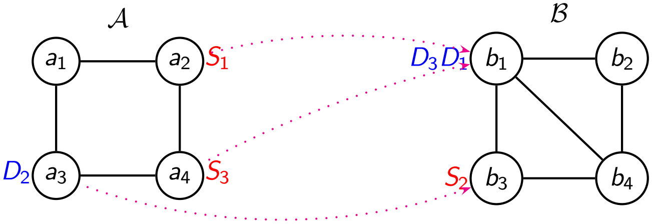
$$\theta : \begin{cases} 1 \mapsto a \\ 2 \mapsto b \\ 3 \mapsto c \end{cases}$$

| $R^{\mathcal{A}}\vert_{\{1,2,3\}}$ | | |
|---|---|---|
| 1 | 2 | 3 |

| $R^{\mathcal{B}}\vert_{\{a,b,c\}}$ | | |
|---|---|---|
| $a$ | $b$ | $c$ |

$\theta$ is a partial isomorphism.

# Partial isomorphisms

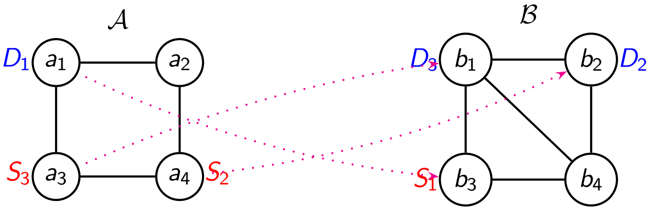



The partial function $\theta : |\mathcal{A}| \to |\mathcal{B}|$ with

$$\theta : \begin{cases} a_2 \mapsto b_1 \\ a_3 \mapsto b_3 \\ a_4 \mapsto b_1 \end{cases}$$

is **not** a partial isomorphism: $\mathcal{A} \models a_2 \neq a_4$, $\mathcal{B} \not\models \theta(a_2) \neq \theta(a_4)$.

# Partial isomorphisms

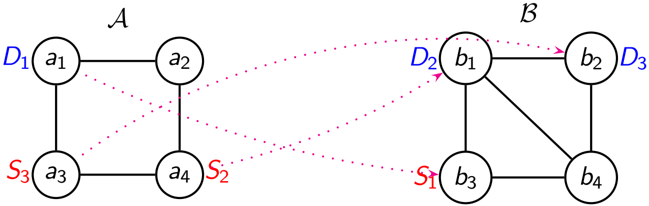



The partial function $\theta : |\mathcal{A}| \to |\mathcal{B}|$ with

$$\theta : \begin{cases} a_1 \mapsto b_3 \\ a_4 \mapsto b_2 \\ a_3 \mapsto b_1 \end{cases}$$

is a partial isomorphism.

# Partial isomorphisms




The partial function $\theta : |\mathcal{A}| \to |\mathcal{B}|$ with

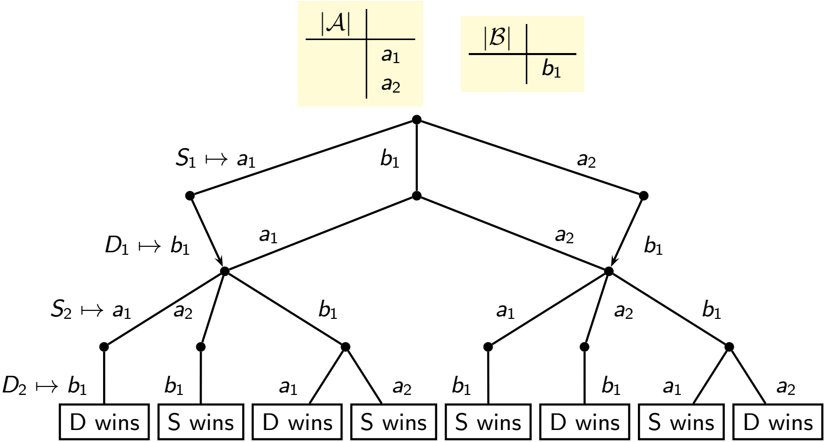
$$\theta : \begin{cases} a_1 \mapsto b_3 \\ a_4 \mapsto b_1 \\ a_3 \mapsto b_2 \end{cases}$$

is not a partial isomorphism: $\mathcal{A} \models E(a_1, a_3)$, $\mathcal{B} \not\models E(\theta(a_1), \theta(a_3))$

# Winning Condition

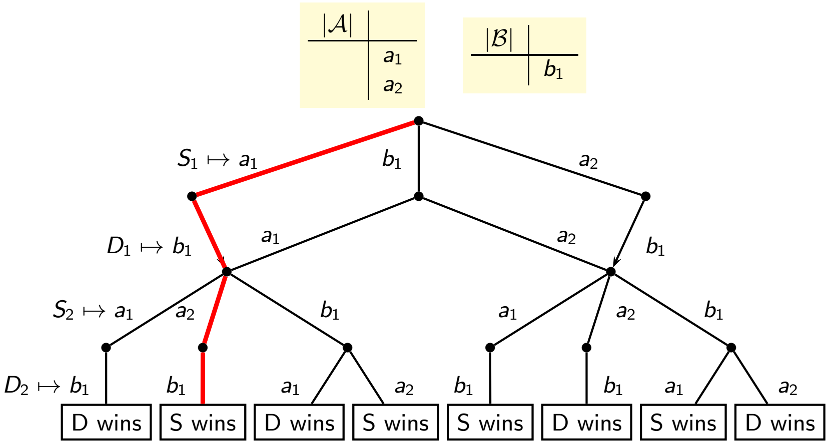
- Duplicator wins a run of the game if the mapping between elements of the two structures defined by the game run is a partial isomorphism.
- Otherwise, Spoiler wins.
- A player has a winning strategy for $k$ moves if s/he can win the $k$-move game no matter how the other player plays.
- Winning strategies can be fully described by finite game trees.
- There is always either a winning strategy for Spoiler or for Duplicator.
- Notation $\mathcal{A} \sim_k \mathcal{B}$: There is a winning strategy for Duplicator for $k$-move games.
- Notation $\mathcal{A} \nsim_k \mathcal{B}$: There is a winning strategy for Spoiler for $k$-move games.

# Game tree of depth 2



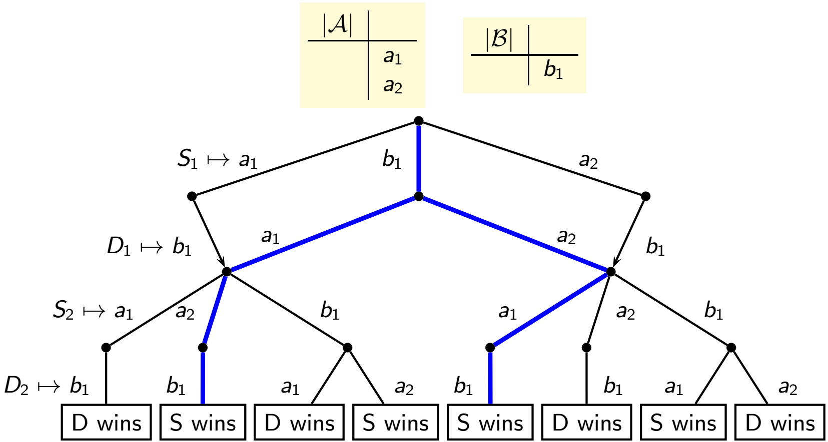

(Here, subtrees are used multiple times to save space – the game tree really is a tree, not a DAG.)

# Game tree of depth 2; Spoiler has a winning strategy



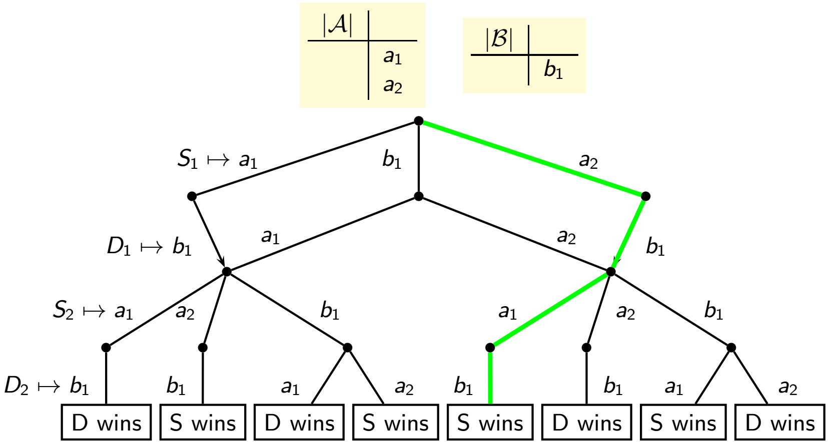

1st winning strategy for Spoiler in two moves ($\mathcal{A} \not\sim_2 \mathcal{B}$)

# Game tree of depth 2; Spoiler has a winning strategy




2nd winning strategy for Spoiler in two moves ($\mathcal{A} \not\sim_2 \mathcal{B}$)

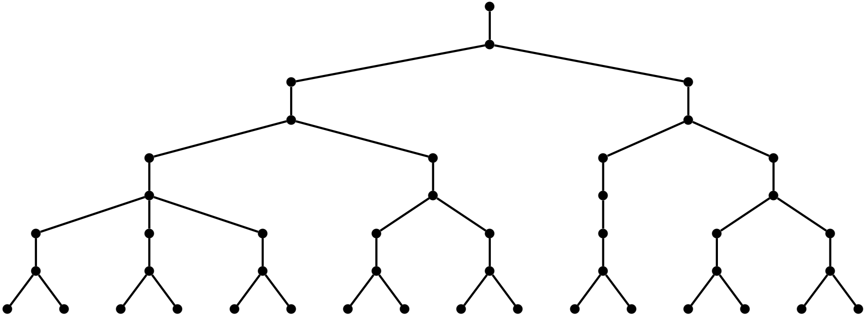
# Game tree of depth 2; Spoiler has a winning strategy




3rd winning strategy for Spoiler in two moves ($\mathcal{A} \not\sim_2 \mathcal{B}$)

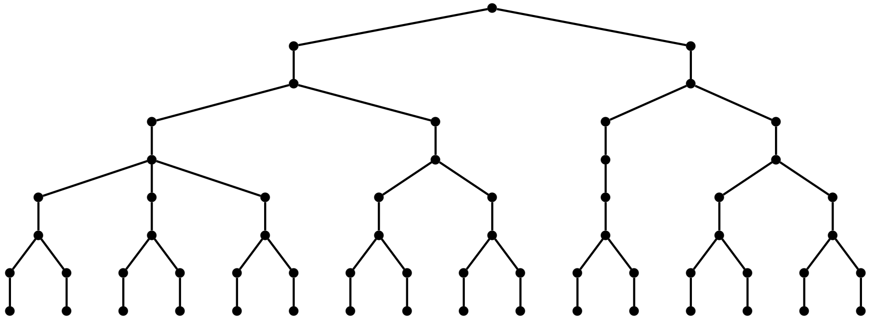
# Schema of a winning strategy for Spoiler

There is a possible move for S such that
for all possible answer moves of D
there is a possible move for S such that
for all possible answer moves of D
⋮
S wins.



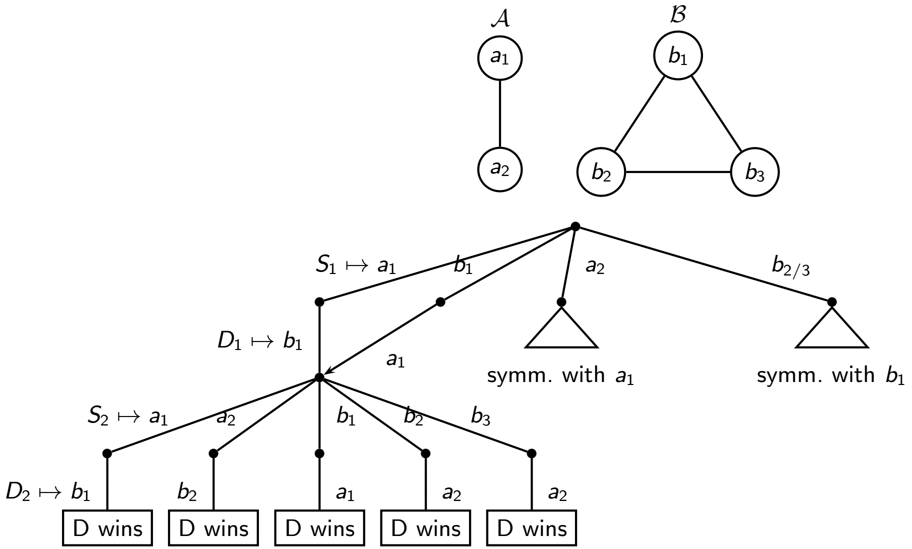
# Schema of a winning strategy for Duplicator

For all possible moves of S
there is a possible answer move for D such that
for all possible moves of S
there is a possible answer move for D such that
⋮
D wins.

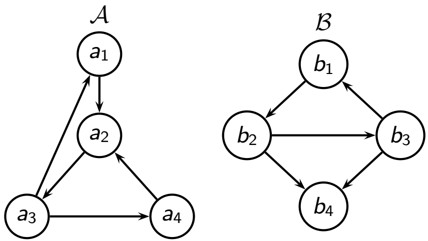


# Examples

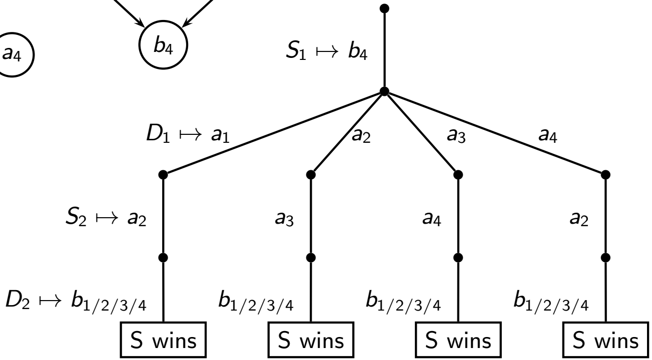
# Example 1: $\mathcal{A} \sim_2 \mathcal{B}$ – Duplicator has a winning strategy



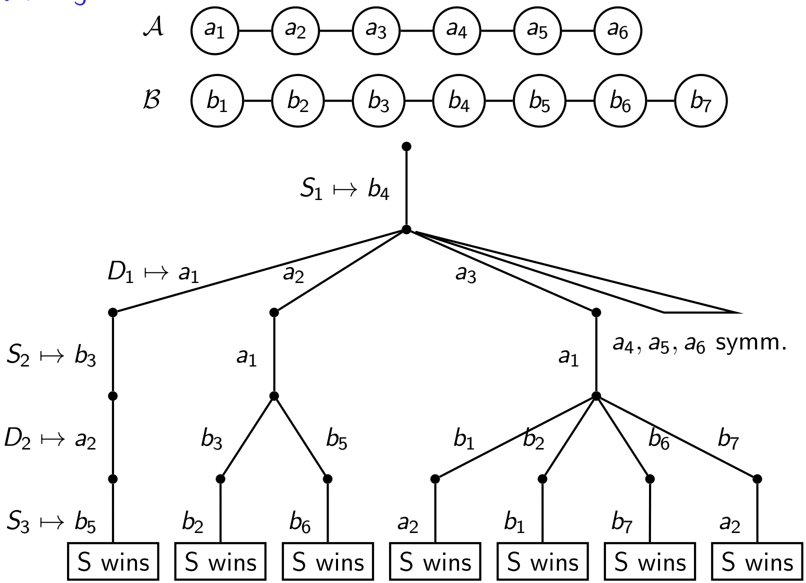
# Example 2: $\mathcal{A} \not\sim_2 \mathcal{B}$ – Spoiler has a winning strategy



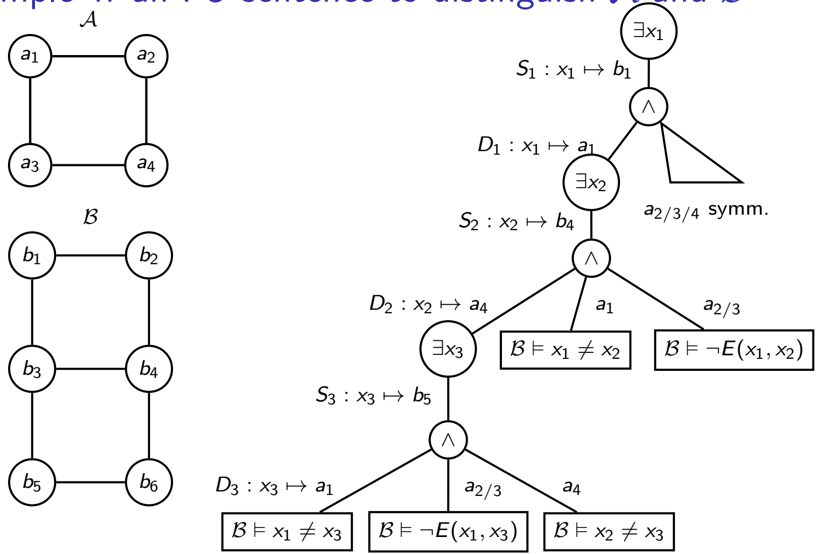

$\mathcal{B} \models \exists x_1 \forall x_2\ \neg E(x_1, x_2)$
$\mathcal{A} \not\models \exists x_1 \forall x_2\ \neg E(x_1, x_2)$



# Example 3: $\mathcal{A} \not\sim_3 \mathcal{B}$



# Example 4: an FO sentence to distinguish $\mathcal{A}$ and $\mathcal{B}$




$\phi = \exists x_1 \exists x_2 \, (\exists x_3 \; x_1 \neq x_3 \wedge \neg E(x_1, x_3) \wedge x_2 \neq x_3) \wedge x_1 \neq x_2 \wedge \neg E(x_1, x_2)$

$\mathcal{B} \models \phi$, $\mathcal{A} \not\models \phi$.

# An FO sentence that distinguishes between $\mathcal{A}$ and $\mathcal{B}$

- Input: a winning strategy for Spoiler.
- We construct a sentence $\phi$ which is true on the structure on which Spoiler puts the first token (this structure is initially the "current structure") and is false on the other structure.
- Spoiler's choice of structure in move $i$ decides the $i$-th quantifier:
  - $\exists x_i$ if $i = 1$ or if Spoiler chooses the same structure that she has chosen in move $i - 1$ and
  - $\neg\exists x_i$ if Spoiler does not choose the same structure as in the previous move. We switch the current structure.
- The alternative answers of Duplicator are combined using conjunctions.
- Each leaf of the strategy tree corresponds to a literal (i.e., a possibly negated atomic formula) that is true on the current structure and false on the other structure. Such a literal exists because Spoiler wins on the leaf, i.e., a mapping is forced that is not a partial isomorphism.

# EF Theorem

# Main theorem

## Definition

We write $\mathcal{A} \equiv_k \mathcal{B}$ for two structures $\mathcal{A}$ and $\mathcal{B}$ if and only if the following is true for all FO sentences $\phi$ of quantifier rank $k$:

$$\mathcal{A} \models \phi \quad \Leftrightarrow \quad \mathcal{B} \models \phi.$$

# Main theorem

## Definition

We write $\mathcal{A} \equiv_k \mathcal{B}$ for two structures $\mathcal{A}$ and $\mathcal{B}$ if and only if the following is true for all FO sentences $\phi$ of quantifier rank $k$:

$$\mathcal{A} \models \phi \quad \Leftrightarrow \quad \mathcal{B} \models \phi.$$

## Theorem (Ehrenfeucht, Fraïssé)

*Given two structures $\mathcal{A}$ and $\mathcal{B}$ and an integer $k$. Then the following statements are equivalent:*

1. $\mathcal{A} \equiv_k \mathcal{B}$, *i.e., $\mathcal{A}$ and $\mathcal{B}$ cannot be distinguished by FO sentences of quantifier rank $k$.*
2. $\mathcal{A} \sim_k \mathcal{B}$, *i.e., Duplicator has a winning strategy for the $k$-move EF game.*

# Proof of the theorem of Ehrenfeucht and Fraïssé

## Proof

- We have provided a method for turning a winning strategy for Spoiler into an FO sentence that distinguishes $\mathcal{A}$ and $\mathcal{B}$.

# Proof of the theorem of Ehrenfeucht and Fraïssé

## Proof

- We have provided a method for turning a winning strategy for Spoiler into an FO sentence that distinguishes $\mathcal{A}$ and $\mathcal{B}$.
- From this it follows immediately that

$$\mathcal{A} \not\sim_k \mathcal{B} \;\Rightarrow\; \mathcal{A} \not\equiv_k \mathcal{B}$$

and thus

$$\mathcal{A} \equiv_k \mathcal{B} \;\Rightarrow\; \mathcal{A} \sim_k \mathcal{B}.$$

# Proof of the theorem of Ehrenfeucht and Fraïssé

## Proof

- We have provided a method for turning a winning strategy for Spoiler into an FO sentence that distinguishes $\mathcal{A}$ and $\mathcal{B}$.
- From this it follows immediately that

$$\mathcal{A} \nsim_k \mathcal{B} \;\Rightarrow\; \mathcal{A} \not\equiv_k \mathcal{B}$$

  and thus

$$\mathcal{A} \equiv_k \mathcal{B} \;\Rightarrow\; \mathcal{A} \sim_k \mathcal{B}.$$

- We still have to prove the other direction ($\mathcal{A} \not\equiv_k \mathcal{B} \;\Rightarrow\; \mathcal{A} \nsim_k \mathcal{B}$).

# Proof of the theorem of Ehrenfeucht and Fraïssé

## Proof

- We have provided a method for turning a winning strategy for Spoiler into an FO sentence that distinguishes $\mathcal{A}$ and $\mathcal{B}$.
- From this it follows immediately that

$$\mathcal{A} \not\sim_k \mathcal{B} \;\Rightarrow\; \mathcal{A} \not\equiv_k \mathcal{B}$$

  and thus

$$\mathcal{A} \equiv_k \mathcal{B} \;\Rightarrow\; \mathcal{A} \sim_k \mathcal{B}.$$

- We still have to prove the other direction ($\mathcal{A} \not\equiv_k \mathcal{B} \;\Rightarrow\; \mathcal{A} \not\sim_k \mathcal{B}$).
- Proof idea: we can construct a winning strategy for Spoiler for the $k$-move EF game from a formula $\phi$ of quantifier rank $k$ with $\mathcal{A} \models \phi$ and $\mathcal{B} \models \neg\phi$. □

# Proof of the theorem of Ehrenfeucht and Fraïssé

## Lemma

*Given a formula* $\phi$ *with* $k = qr(\phi)$ *and* $free(\phi) = \{x_1, \ldots, x_\ell\}$ *for* $\ell \geq 0$.
*If* $\mathcal{A} \models \phi[a_{i_1}, \ldots, a_{i_\ell}]$ *and* $\mathcal{B} \models (\neg\phi)[b_{j_1}, \ldots, b_{j_\ell}]$ *then Spoiler has a winning strategy in the* $k + \ell$ *round EF game starting with* $a_{i_1} \mapsto b_{j_1}, \ldots, a_{i_\ell} \mapsto b_{j_\ell}$.

# Proof of the theorem of Ehrenfeucht and Fraïssé

## Lemma

*Given a formula $\phi$ with $k = qr(\phi)$ and $free(\phi) = \{x_1, \ldots, x_\ell\}$ for $\ell \geq 0$.*
*If $\mathcal{A} \vDash \phi[a_{i_1}, \ldots, a_{i_\ell}]$ and $\mathcal{B} \vDash (\neg\phi)[b_{j_1}, \ldots, b_{j_\ell}]$ then Spoiler has a winning strategy in the $k + \ell$ round EF game starting with $a_{i_1} \mapsto b_{j_1}, \ldots, a_{i_\ell} \mapsto b_{j_\ell}$.*

## Proof

W.l.o.g., we assume that $\phi$ contains no universal quantification. This can be easily achieved by replacing every subformula of the form $\forall x\ \psi$ in $\phi$ by $\neg\exists x\ \neg\psi$. The proof proceeds by structural induction on $\phi$:

- If $\phi$ is an atomic formula, then $k = \mathrm{qr}(\phi) = 0$. Clearly, in this case, the Spoiler wins in $\ell$ rounds with $a_{i_1} \mapsto b_{j_1}, \ldots, a_{i_\ell} \mapsto b_{j_\ell}$.
- If $\phi = \neg\psi$, then we have $\mathcal{B} \vDash \psi[b_{j_1}, \ldots, b_{j_\ell}]$ and $\mathcal{A} \vDash (\neg\psi)[a_{i_1}, \ldots, a_{i_\ell}]$. Hence, by the induction hypothesis, the Spoiler has a winning strategy in the $k + \ell$ round EF game starting with $b_{j_1} \mapsto a_{i_1}, \ldots, b_{j_\ell} \mapsto a_{i_\ell}$.

# Proof of the theorem of Ehrenfeucht and Fraïssé

## Proof (continued)

- If $\phi = \psi_1 \wedge \psi_2$ then $\neg\phi = (\neg\psi_1) \vee (\neg\psi_2)$. By $\mathcal{B} \models (\neg\phi)[b_{j_1}, \ldots, b_{j_\ell}]$, for at least one $i \in \{1, 2\}$, $\mathcal{B} \models (\neg\psi_i)[b_{j_1}, \ldots, b_{j_\ell}]$ holds. Moreover, by $\mathcal{A} \models \phi[a_{i_1}, \ldots, a_{i_\ell}]$ also $\mathcal{A} \models \psi_i[a_{i_1}, \ldots, a_{i_\ell}]$ holds. Hence, by the induction hypothesis, the Spoiler has a winning strategy in the $k + \ell$ round EF game starting with $a_{j_1} \mapsto b_{i_1}, \ldots, a_{j_\ell} \mapsto b_{i_\ell}$.
- If $\phi = \psi_1 \vee \psi_2$ then $\neg\phi = (\neg\psi_1) \wedge (\neg\psi_2)$; as above.
- $\phi = \exists x_{\ell+1}\ \psi$: There exists an element $a_{i_{\ell+1}}$ such that $\mathcal{A} \models \psi[a_{i_1}, \ldots, a_{i_{\ell+1}}]$ but for all $b_{j_{\ell+1}}$, $\mathcal{B} \models (\neg\psi)[b_{j_1}, \ldots, b_{j_{\ell+1}}]$. If the induction hypothesis holds for $\psi$ then it also holds for $\phi$. □

# Proof of the theorem of Ehrenfeucht and Fraïssé

## Proof (continued)

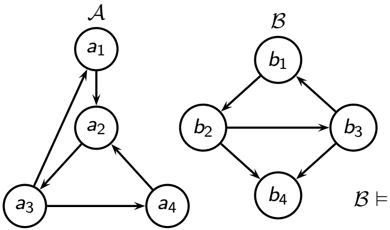
- If $\phi = \psi_1 \wedge \psi_2$ then $\neg\phi = (\neg\psi_1) \vee (\neg\psi_2)$. By $\mathcal{B} \models (\neg\phi)[b_{j_1}, \ldots, b_{j_\ell}]$, for at least one $i \in \{1, 2\}$, $\mathcal{B} \models (\neg\psi_i)[b_{j_1}, \ldots, b_{j_\ell}]$ holds. Moreover, by $\mathcal{A} \models \phi[a_{i_1}, \ldots, a_{i_\ell}]$ also $\mathcal{A} \models \psi_i[a_{i_1}, \ldots, a_{i_\ell}]$ holds. Hence, by the induction hypothesis, the Spoiler has a winning strategy in the $k + \ell$ round EF game starting with $a_{j_1} \mapsto b_{i_1}, \ldots, a_{j_\ell} \mapsto b_{i_\ell}$.
- If $\phi = \psi_1 \vee \psi_2$ then $\neg\phi = (\neg\psi_1) \wedge (\neg\psi_2)$; as above.
- $\phi = \exists x_{\ell+1}\, \psi$: There exists an element $a_{i_{\ell+1}}$ such that $\mathcal{A} \models \psi[a_{i_1}, \ldots, a_{i_{\ell+1}}]$ but for all $b_{j_{\ell+1}}$, $\mathcal{B} \models (\neg\psi)[b_{j_1}, \ldots, b_{j_{\ell+1}}]$. If the induction hypothesis holds for $\psi$ then it also holds for $\phi$. □

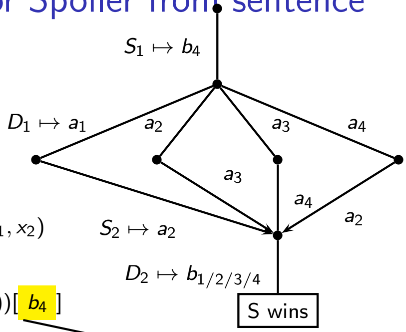
From the above lemma, by setting $\ell = 0$, we immediately get:

## Lemma

*If $\mathcal{A} \not\equiv_k \mathcal{B}$ then $\mathcal{A} \nsim_k \mathcal{B}$.*

# Construction: Winning strategy for Spoiler from sentence




$\mathcal{B} \models \exists x_1 \forall x_2\ \neg E(x_1, x_2)$

$\mathcal{B} \models (\forall x_2\ \neg E(x_1, x_2))[\,b_4\,]$

$\mathcal{B} \models (\neg E(x_1, x_2))[b_4, b_1]$ $\mathcal{B} \models (\neg E(x_1, x_2))[b_4, b_2]$ $\mathcal{B} \models (\neg E(x_1, x_2))[b_4, b_3]$ $\mathcal{B} \models (\neg E(x_1, x_2))[b_4, b_4]$

$\mathcal{A} \models \forall x_1 \exists x_2\ E(x_1, x_2)$

$\mathcal{A} \models (\exists x_2\ E(x_1, x_2))[a_1]$ $\mathcal{A} \models (\exists x_2\ E(x_1, x_2))[a_2]$ $\mathcal{A} \models (\exists x_2\ E(x_1, x_2))[a_3]$ $\mathcal{A} \models (\exists x_2\ E(x_1, x_2))[a_4]$

$\mathcal{A} \models E(x_1, x_2)[\,a_1, a_2\,]$ $\mathcal{A} \models E(x_1, x_2)[\,a_2, a_3\,]$ $\mathcal{A} \models E(x_1, x_2)[\,a_3, a_4\,]$ $\mathcal{A} \models E(x_1, x_2)[\,a_4, a_2\,]$

# Inexpressibility proofs

# Inexpressibility proofs

- Expressibility of a query in FO means that there is an FO formula equivalent to that query;
- if there is such a formula, it must have some quantifier rank.
- We thus get the following methodology for proving inexpressibility:

# Inexpressibility proofs

- Expressibility of a query in FO means that there is an FO formula equivalent to that query;
- if there is such a formula, it must have some quantifier rank.
- We thus get the following methodology for proving inexpressibility:

## Theorem (Methodology theorem)

*Given a Boolean query $Q$. There is* **no** *FO sentence that expresses $Q$ if and only if there are, for each $k$, structures $\mathcal{A}_k$, $\mathcal{B}_k$ such that*

- $\mathcal{A}_k \models Q$,
- $\mathcal{B}_k \not\models Q$ *and*
- $\mathcal{A}_k \sim_k \mathcal{B}_k$.

# Inexpressibility proofs

- Expressibility of a query in FO means that there is an FO formula equivalent to that query;
- if there is such a formula, it must have some quantifier rank.
- We thus get the following methodology for proving inexpressibility:

## Theorem (Methodology theorem)

*Given a Boolean query $Q$. There is* **no** *FO sentence that expresses $Q$ if and only if there are, for each $k$, structures $\mathcal{A}_k$, $\mathcal{B}_k$ such that*

- $\mathcal{A}_k \models Q$,
- $\mathcal{B}_k \not\models Q$ *and*
- $\mathcal{A}_k \sim_k \mathcal{B}_k$.

Thus, EF games provide a complete methodology for constructing inexpressibility proofs. To prove inexpressibility, we only have to

- construct suitable structures $\mathcal{A}_k$ and $\mathcal{B}_k$ and
- prove that $\mathcal{A}_k \sim_k \mathcal{B}_k$. (This is usually the difficult part.)

# Example: Inexpressibility of the parity query

## Definition (parity query)

Given a structure $\mathcal{A}$ with empty schema (i.e., only $|\mathcal{A}|$ is given). Question: Does $|\mathcal{A}|$ have an even number of elements?

- Construction of the structures $\mathcal{A}_n$ and $\mathcal{B}_n$ for arbitrary $n$:

$$|\mathcal{A}_n| := \{a_1, \ldots, a_n\} \qquad |\mathcal{B}_n| := \{b_1, \ldots, b_{n+1}\}$$

## Lemma

*$\mathcal{A}_n \sim_k \mathcal{B}_n$ for all $k \leq n$.*

(This is shown on the next slide.)

# Example: Inexpressibility of the parity query

## Definition (parity query)

Given a structure $\mathcal{A}$ with empty schema (i.e., only $|\mathcal{A}|$ is given). Question: Does $|\mathcal{A}|$ have an even number of elements?

- Construction of the structures $\mathcal{A}_n$ and $\mathcal{B}_n$ for arbitrary $n$:

$$|\mathcal{A}_n| := \{a_1, \ldots, a_n\} \qquad |\mathcal{B}_n| := \{b_1, \ldots, b_{n+1}\}$$

## Lemma

*$\mathcal{A}_n \sim_k \mathcal{B}_n$ for all $k \leq n$.*

(This is shown on the next slide.)

- On the other hand, $\mathcal{A}_n \models$ Parity if and only if $\mathcal{B}_n \not\models$ Parity.
- It thus follows from the methodology theorem that parity is not expressible in FO.

# Example: Inexpressibility of the parity query

**Lemma**

$\mathcal{A}_n \sim_k \mathcal{B}_n$ *for all* $k \leq n$.

# Example: Inexpressibility of the parity query

## Lemma

*$\mathcal{A}_n \sim_k \mathcal{B}_n$ for all $k \leq n$.*

## Proof

We construct a winning strategy for Duplicator. This time no strategy trees are explicitly shown, but a general construction is given.
We handle the case in which Spoiler plays on $\mathcal{A}_n$. The other direction is analogous. If $S_i \mapsto a$ then

- $D_i \mapsto b$ where $b$ is a new element of $|\mathcal{B}_n|$ if $a$ has not been played on yet (=no token was put on it);
- If, for some $j < i$, $S_j \mapsto a, D_j \mapsto b'$ or $S_j \mapsto b', D_j \mapsto a$ was played then $D_i \mapsto b'$.

Over $k$ moves, we only construct partial isomorphisms in this way and obtain a winning strategy for Duplicator. □

# Undirected Paths

## Theorem

*Let $L_1$, $L_2$ be undirected paths of length $\geq 2^k$. Then $L_1 \sim_k L_2$ holds.*

# Undirected Paths

## Theorem

*Let $L_1$, $L_2$ be undirected paths of length $\geq 2^k$. Then $L_1 \sim_k L_2$ holds.*

## Proof Idea

- Consider the nodes in $L_1$ and $L_2$ arranged from left to right, s.t. we have a linear order on the nodes.
- Add nodes "min" on the left and "max" on the right of each path.
- For every $i \in \{0, \ldots, k\}$, consider the $i$-round EF-game and assume that before the actual game, the additional nodes "min" and "max" are played in the two graphs.
- Hence, after $i$ moves, the players have chosen vectors $\vec{a} = (a_{-1}, a_0, a_1, \ldots, a_i)$ in $L_1$ and $\vec{b} = (b_{-1}, b_0, b_1, \ldots, b_i)$ in $L_2$ with $a_{-1} = b_{-1} =$ "min" and $a_0 = b_0 =$ "max".
- As usual, we define the distance $d(u, v)$ between two nodes $u$ and $v$ as the length of the shortest path between $u$ and $v$.

## Proof (continued)

A winning strategy for the Duplicator can be obtained as follows:
The Duplicator can play in such a way that for every $j, l \in \{-1, \ldots, i\}$, the following conditions hold:

1. if $d(a_j, a_l) < 2^{k-i}$, then $d(a_j, a_l) = d(b_j, b_l)$;
2. if $d(a_j, a_l) \geq 2^{k-i}$, then $d(b_j, b_l) \geq 2^{k-i}$;
3. $a_j \leq a_l$ if and only if $b_j \leq b_l$

## Proof (continued)

A winning strategy for the Duplicator can be obtained as follows:
The Duplicator can play in such a way that for every $j, l \in \{-1, \ldots, i\}$, the following conditions hold:

1. if $d(a_j, a_l) < 2^{k-i}$, then $d(a_j, a_l) = d(b_j, b_l)$;
2. if $d(a_j, a_l) \geq 2^{k-i}$, then $d(b_j, b_l) \geq 2^{k-i}$;
3. $a_j \leq a_l$ if and only if $b_j \leq b_l$

The claim is proved by induction on $i$:
$i = 0$. Clear. In particular, we have $d(a_{-1}, a_0) \geq 2^{k-0}$ and $d(b_{-1}, b_0) \geq 2^{k-0}$.

## Proof (continued)

A winning strategy for the Duplicator can be obtained as follows:
The Duplicator can play in such a way that for every $j, l \in \{-1, \dots, i\}$, the following conditions hold:

1. if $d(a_j, a_l) < 2^{k-i}$, then $d(a_j, a_l) = d(b_j, b_l)$;
2. if $d(a_j, a_l) \geq 2^{k-i}$, then $d(b_j, b_l) \geq 2^{k-i}$;
3. $a_j \leq a_l$ if and only if $b_j \leq b_l$

The claim is proved by induction on $i$:
$i = 0$. Clear. In particular, we have $d(a_{-1}, a_0) \geq 2^{k-0}$ and $d(b_{-1}, b_0) \geq 2^{k-0}$.

$i \to i+1$. Suppose the spoiler makes the $(i+1)$st move in $L_1$.
(the case of $L_2$ is symmetric.)
Case 1. $a_{i+1} = a_j$ for some $j$. Then the Duplicator chooses $b_{i+1} = b_j$.
Case 2. $a_{i+1}$ is in the interval $a_j$ and $a_l$ for some $j, l$.

## Proof (continued)

Case 2.1. $a_{i+1}$ is "close to" $a_j$, i.e., $d(a_j, a_{i+1}) < 2^{k-i-1}$.
Then the Duplicator chooses $b_{i+1}$ in the interval $b_j$ and $b_l$ with $d(b_j, b_{i+1}) = d(a_j, a_{i+1})$.

## Proof (continued)

Case 2.1. $a_{i+1}$ is "close to" $a_j$, i.e., $d(a_j, a_{i+1}) < 2^{k-i-1}$.
Then the Duplicator chooses $b_{i+1}$ in the interval $b_j$ and $b_l$ with $d(b_j, b_{i+1}) = d(a_j, a_{i+1})$.

Case 2.2. $a_{i+1}$ is "close to" $a_l$, i.e., $d(a_{i+1}, a_l) < 2^{k-i-1}$.
Then the Duplicator chooses $b_{i+1}$ in the interval $b_j$ and $b_l$ with $d(b_{i+1}, b_l) = d(a_{i+1}, a_l)$.

## Proof (continued)

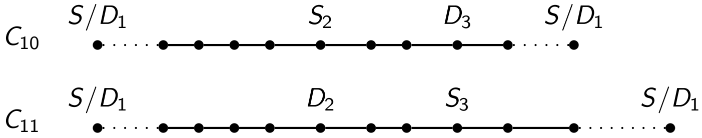
Case 2.1. $a_{i+1}$ is "close to" $a_j$, i.e., $d(a_j, a_{i+1}) < 2^{k-i-1}$.
Then the Duplicator chooses $b_{i+1}$ in the interval $b_j$ and $b_l$ with $d(b_j, b_{i+1}) = d(a_j, a_{i+1})$.

Case 2.2. $a_{i+1}$ is "close to" $a_l$, i.e., $d(a_{i+1}, a_l) < 2^{k-i-1}$.
Then the Duplicator chooses $b_{i+1}$ in the interval $b_j$ and $b_l$ with $d(b_{i+1}, b_l) = d(a_{i+1}, a_l)$.

Case 2.3. $a_{i+1}$ is "far away from" both $a_j$ and $a_l$, i.e., $d(a_j, a_{i+1}) \geq 2^{k-i-1}$ and $d(a_{i+1}, a_l) \geq 2^{k-i-1}$.
Then the Duplicator chooses $b_{i+1}$ in the middle between $b_j$ and $b_l$. □

# Cycles

- (Isolated) undirected cycles $C_n$: Graphs with nodes $\{v_1, \ldots, v_n\}$ and edges $\{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$.
- After the first move, there is one distinguished node in the cycle, the one with token $S_1$ or $D_1$ on it.
- We can treat this cycle like a path obtained by cutting the cycle at the distinguished node.




- Theorem. If $n \geq 2^k$, then $C_n \sim_k C_{n+1}$.

# 2-colorability

## Definition

2-colorability: Given a graph, is there a function that maps each node to either "red" or "green" such that no two adjacent nodes have the same color?

## Theorem

*2-colorability is not expressible in FO.*

# 2-colorability

## Definition

2-colorability: Given a graph, is there a function that maps each node to either "red" or "green" such that no two adjacent nodes have the same color?

## Theorem

*2-colorability is not expressible in FO.*

## Proof Sketch

For each $k$,

- $\mathcal{A}_k$: $C_{2^k}$, the cycle of length $2^k$.
- $\mathcal{B}_k$: $C_{2^k+1}$, the cycle of length $2^k + 1$.
- $\mathcal{A}_k \sim_k \mathcal{B}_k$.
- However, a cycle $C_n$ of length $n$ is 2-colorable iff $n$ is even.

Inexpressibility follows from the EF methodology theorem. □

# Acyclicity

From now on, “very long/large” means simply $2^k$.

**Theorem**

*Acyclicity is not expressible in FO.*

# Acyclicity

From now on, “very long/large” means simply $2^k$.

## Theorem

*Acyclicity is not expressible in FO.*

## Proof Sketch

- $\mathcal{A}_k$: a very long path.
- $\mathcal{B}_k$: a very long path plus (disconnected from it) a very large cycle.
- $\mathcal{A}_k \sim_k \mathcal{B}_k$. □

# Graph reachability

## Theorem

*Graph reachability from $a$ to $b$ is not expressible in FO.*

$a$, $b$ are constants or are given by an additional unary relation with two entries.

## Proof Sketch

- $\mathcal{A}_k$: a very large cycle in which the nodes $a$ and $b$ are maximally distant.
- $\mathcal{B}_k$: two very large cycles; $a$ is a node of the first cycle and $b$ a node of the second.
- $\mathcal{A}_k \sim_k \mathcal{B}_k$. □

Remark. The same structures $\mathcal{A}_k$, $\mathcal{B}_k$ can be used to show that connectedness of a graph is not expressible in FO.

# Linear Orders

## Definition

A linear order is a structure $L = (|L|, <)$, where $<$ is a total order on the elements in $|L|$, that is, $<$ has the following properties:

- irreflexive: $\forall x\ \neg(x < x)$;
- transitive: $\forall x \forall y \forall z\ (x < y \land y < z) \rightarrow (x < z)$;
- total: $\forall x \forall y\ (x \neq y) \rightarrow (x < y \lor y < x)$.

## Theorem

*Let $L_1$ and $L_2$ be two (finite) linear orders of length at least $2^k$. Then $L_1 \sim_k L_2$ holds.*

Proof. By exactly the same idea as for undirected paths.

## Theorem

*The parity query on linear orders is not expressible in FO.*

# Further Examples

## Theorem

*The following Boolean queries are not expressible in FO:*

- *Hamiltonicity (does the graph have a Hamilton cycle);*
- *Eulerian Graph (does the graph have a Eulerian cycle, i.e., a round trip that visits each edge of the graph exactly once);*
- *$k$-Colorability for arbitrary $k \geq 2$;*
- *Existence of a clique of size $\geq n/2$ (with $n =$ number of vertices).*

# Learning Objectives

- Rules of EF game
- Winning condition and winning strategies of EF games
- EF Theorem and its proof
- Inexpressibility proofs using the Methodology theorem

# Literature


- Phokion Kolaitis, "Combinatorial Games in Finite Model Theory":
  http://www.cse.ucsc.edu/∼kolaitis/talks/essllif.ps (Slides 1–40)
- Abiteboul, Hull, Vianu, "Foundations of Databases", Addison-Wesley 1994. Chapter 17.2.
- Libkin, "Elements of Finite Model Theory", Springer 2004. Chapter 3.
- Ebbinghaus, Flum, "Finite Model Theory", Springer 1999. Chapter 2.1–2.3.