Database Theory

Unit 2 — Datalog

Matthias Lanzinger, WS2024/25, partially based on Arenas, et al., “Database Theory”

TU WIEN Informatics

Motivation

✦Keep the logical foundations for relational calculus to give the language a strong theoretical foundation.

✦Have a simple and intuitively understandable language that can be used by domain experts rather than programmers.

✦Support recursion to allow powerful queries for hierarchical data or graph-like structures.

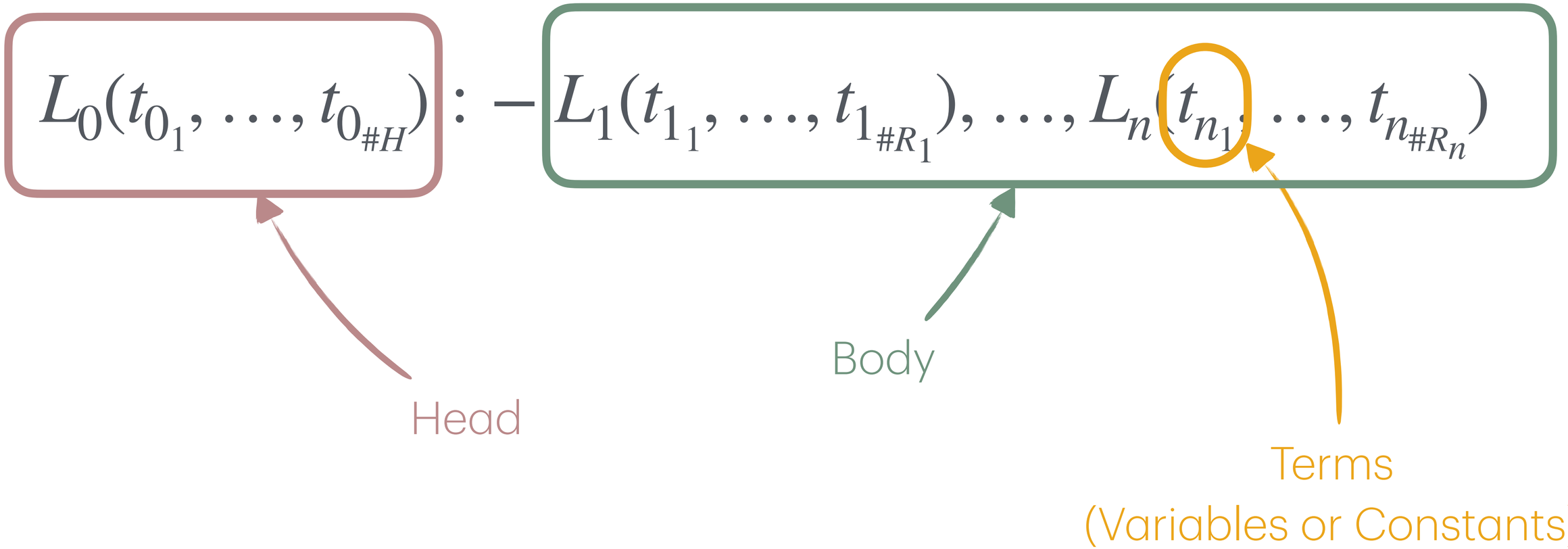
✦Combine all of these with efficiency in evaluation.

Syntax of Datalog

Datalog Rule

$$L_0(t_{0_1}, \ldots, t_{0_{\#H}}) \overset{\text{“If”}}{:-} L_1(t_{1_1}, \ldots, t_{1_{\#R_1}}), \ldots, L_n(t_{n_1}, \ldots, t_{n_{\#R_n}})$$

Datalog Rule



Datalog Rule

$$L_0(t_{0_1}, \ldots, t_{0_{\#H}}) :- L_1(t_{1_1}, \ldots, t_{1_{\#R_1}}), \ldots, L_n(t_{n_1}, \ldots, t_{n_{\#R_n}})$$

IDB vs EDB

We partition relation names into to intensional and extensional predicates (IDBs and EDBs).

A relation name is an IDB if it occurs in the head of a rule.

Intuitively, EDBs are those predicates that come from the database.

Datalog Rule

$$L_0(t_{0_1}, \ldots, t_{0_{\#H}}) :- L_1(t_{1_1}, \ldots, t_{1_{\#R_1}}), \ldots, L_n(t_{n_1}, \ldots, t_{n_{\#R_n}})$$

$vars(L)$:= the set of variables in literal L

$vars(r)$:= the set of all variables in the rule

$vars(body(r))$:= the set of all variables in the rule body

Datalog Rule

$$L_0(t_{0_1}, \ldots, t_{0_{\#H}}) :- L_1(t_{1_1}, \ldots, t_{1_{\#R_1}}), \ldots, L_n(t_{n_1}, \ldots, t_{n_{\#R_n}})$$

Important Restrictions

1. Variables in the head must occur in the body,
   i.e,. $vars(L_0) \subseteq vars(body(r))$.
2. EDB predicates cannot occur in the head of rules.

Datalog Program

A Datalog program is simply a set of Datalog rules.

A Datalog query is a pair (Π, P) where Π is a Datalog program and P is an IDB relation name.

Example Program:

$$\begin{aligned} CoAuthor(y, x) &:- Write(x, p),\ Write(y, p) \\ HasEN(x) &:- HasEN(y),\ CoAuthor(y, x) \\ HasEN(\text{Erdos})\ &. \end{aligned}$$

Syntactic convention for empty body

$HasEN$ and $CoAuthor$ are IDBs. When used with a database we expect the EDB relation $Write$ to be defined in the database.

Operational Semantics

Some Notation

- In this context it will be convenient to introduce new notation to check whether a specific tuple is in a specific relation of a database D.
- For relation name R, we write $R(c_1, \ldots, c_\ell) \in D$ to mean that the tuple $(c_1, \ldots, c_\ell)$ is in R^D

In the same sense we will also sometimes define databases via sets of atoms, e.g., the set

$$\{R(1,2), R(2,3), S(a,b)\}$$

Defines the following database:

Relation R

A	B
1	2
2	3

Relation S

X	Y
A	B

Homomorphism

$$L_0(t_{0_1}, \ldots, t_{0_{\#H}}) :- L_1(t_{1_1}, \ldots, t_{1_{\#R_1}}), \ldots, L_n(t_{n_1}, \ldots, t_{n_{\#R_n}})$$

We want to talk about whether the body of rule r maps into a database D.

That is, we are interested in assignment h of all terms in the body to constants, such that for every $1 \leq i \leq n$ we have $L_i(h(t_{i_1}), \ldots, h(t_{i_{\#R_i}})) \in D$.

Since we don't want to change terms in the body that are already constants, we restrict h such that $h(c) = c$ for every constant $c \in Dom$.

We call such a function h a homomorphism from $body(r)$ into D.

Immediate Consequence

Let Π be a Datalog program and D a database.

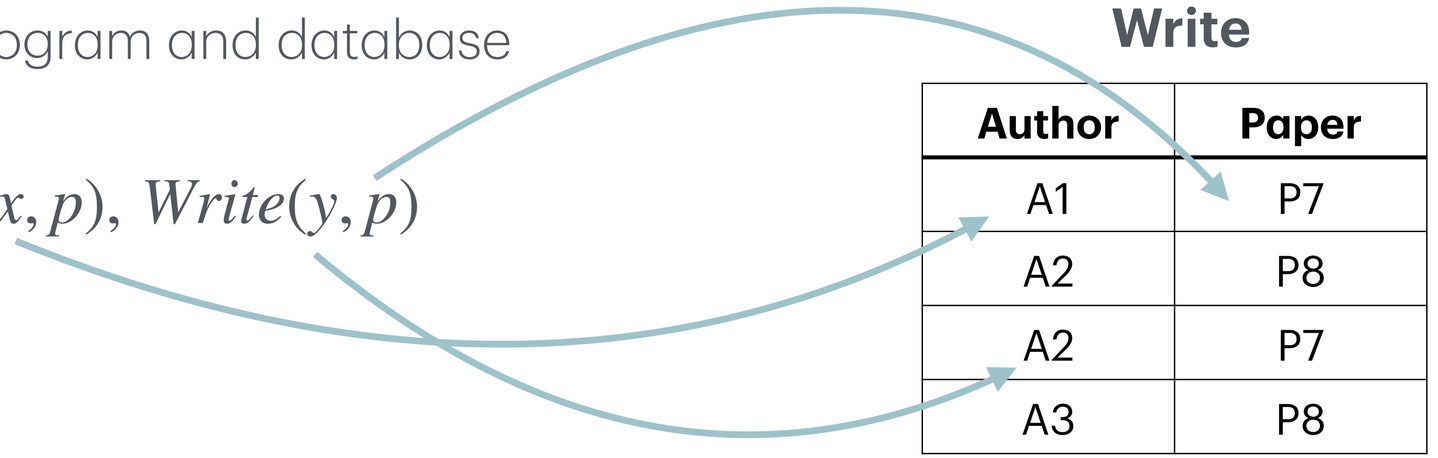
A relational atom $R(c_1, \ldots, c_\ell)$ is an immediate consequence of Π and D if:

- $R(c_1, \ldots, c_\ell) \in D$
- There exists a rule $r = R(y_1, \ldots, y_\ell) :- L_1(\bar{x_1}), \ldots, L_n(\bar{x_n})$ such that there is a homomorphism h from $body(r)$ to D with $h(y_i) = c_i$ for $1 \leq i \leq \ell$

Immediate Consequence — Example

Say we have the following program and database

$$CoAuthor(y, x) :- \quad Write(x, p),\ Write(y, p)$$



Write

Author	Paper
A1	P7
A2	P8
A2	P7
A3	P8

There is a homomorphism $x \mapsto A1, y \mapsto A2, p \mapsto P7$.

Hence, $CoAuthor(A2, A1)$ is an immediate consequence of this program and database.

Immediate Consequence Operator

Intuitively, Datalog semantics add all immediate consequences to the databases until there are no more new immediate consequences.

The immediate consequence operator T_Π that maps a database to a new database as follows:

$$T_\Pi(D) = \{\ R(\bar{a}) \mid R(\bar{a}) \text{ is an immediate consequence of } \Pi \text{ and } D\ \}$$

We call D a fixpoint of T_Π if $T_\Pi(D) = D$.

Datalog Semantics

Starting from a Datalog program Π and database D we can now define the result of applying the program to the database as follows:

$$\text{Let} \quad T^0_\Pi(D) = D \quad \text{and} \quad T^{i+1}_\Pi(D) = T_\Pi(T^i_\Pi(D))$$

$$\text{and } T^\infty_\Pi = \bigcup_{i \geq 0} T^i_\Pi(D)$$

We write $\Pi_{\text{fp}}(D)$ for result of program Π applied to D and define it as

$$\Pi_{\text{fp}}(D) := T^\infty_\Pi(D)$$

Example

Recall our example program

$$CoAuthor(y, x) :- Write(x, p), Write(y, p)$$
$$HasEN(x) :- HasEN(y), CoAuthor(y, x)$$
$$HasEN(\text{Erdos}).$$

Write

Author	Paper
A1	P7
A2	P8
A2	P7
Erdos	P8

$$T^1_\Pi(D) = D \cup \{ HasEN(Erdos), CoAuthor(Erdos, A2), CoAuthor(A2, Erdos), CoAuthor(A1, A2), CoAuthor(A2, A1) \}.$$

Now there is a new homomorphism from the second rule into $T^1_\Pi(D)$: $y \mapsto Erdos, x \mapsto A2$

$$T^2_\Pi(D) = T^1_\Pi(D) \cup \{HasEN(A2)\} \qquad T^3_\Pi(D) = T^2_\Pi(D) \cup \{HasEN(A1)\}$$

Fixpoint!

Finiteness of $T_{\Pi}^{\infty}(D)$

Observe that if $R(c_1, \ldots, c_\ell)$ is an immediate consequence of Π and D, then every c_i must either come from the database or be a fixed part of the program.

Let $Dom(\Pi, D)$ be the set of all constants that occur in at least one of Π and D.
Since both Π and D are finite, so is $Dom(\Pi, D)$ $\Rightarrow$ Finitely many $R(c_1, \ldots, c_\ell)$ to add.

Proposition

Let r be the number of IDBs in Π and a be their maximum arity,
Then $T_{\Pi}^{\infty}(D) = T_{\Pi}^{n}(D)$ where $n \leq r \cdot |Dom(\Pi, D)|^a$.

Datalog Semantics

The semantics of a Datalog query $q = (\Pi, R)$ are as follows:

$$q(D) := \{(a_1, \ldots, a_{\#R}) \in Dom^{\#R} \mid R(a_1, \ldots, a_{\#R}) \in \Pi_{\text{fp}}(D)\}$$

In our running example $q = (\Pi, HasEN)$ returns the set of all authors with finite Erdos number.

$$\begin{aligned} CoAuthor(y, x) &:- Write(x, p),\ Write(y, p) \\ HasEN(x) &:- HasEN(y),\ CoAuthor(y, x) \\ HasEN(\text{Erdos})&\ . \end{aligned}$$

Logical Semantics

FO Theories

- Recall, a set Φ of first-order sentences is called a (first-order) theory.
- A database D is a model of theory Φ
  if $D \models \varphi$ for every $\varphi \in \Phi$.
- First-order formulas and databases are always defined over some schema $\mathcal{S}$.
  We will not explicitly mention the schema every time here. Implicitly we always talk about databases and theories over a single shared schema here.

Datalog as a FO Theory

For Datalog rule $r = R(\bar{x}) :- L_1(\bar{x_1}), \ldots, L_n(\bar{x_n})$ we define

$$\varphi_r := \forall z_1, z_2, \ldots z_m \,.\, (L_1(\bar{x_1}) \wedge \cdots L_n(\bar{x_n})) \rightarrow R(\bar{x})$$

where $\{z_1, \ldots, z_m\} = vars(r)$

Example

$$r = HasEN(x) :- HasEN(y), CoAuthor(y, x)$$

$$vars(r) = \{x, y\}$$

$$\varphi_r = \forall x, y \,(HasEN(y) \wedge CoAuthor(x, y)) \rightarrow HasEN(x)$$

Datalog as a FO Theory

For Datalog program Π we define the theory

$$\Phi_\Pi \quad := \quad \{ \varphi_r \mid r \in \Pi \}$$

The theory captures logically what we mean by the rules of the program.
A model D' such that $D' \supseteq D$ and $D' \models \Phi_\Pi$ makes all our rules hold true and is therefore at first sight a reasonable result of evaluating Π on D.

Problem:

there are always infinitely many such D'!

Minimal Models

For program Π and database D we define the set of minimal models:

$$\textbf{MinMod}(\Pi, D) := \{\, D' \mid D \subseteq D', D' \text{ is a } \subseteq \text{-minimal model of } \Phi_\Pi \,\}$$

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Example

$$\varphi_r = \forall x, y\,(HasEN(y) \wedge CoAuthor(x, y)) \rightarrow HasEN(x)$$

$$D = \{HasEN(\text{Erdos}), CoAuthor(\text{Erdos}, A1), CoAuthor(A3, A4)\}$$

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Is $D \cup \{HasEN(A1), HasEN(A3), HasEN(A4)\}$ a minimal model of φ_r?

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$$D = \{HasEN(\text{Erdos}), CoAuthor(\text{Erdos}, A1), CoAuthor(A3, A4)\}$$

Is $D \cup \{HasEN(A1), HasEN(A3), HasEN(A4)\}$ a model of φ_r?

No — It is a model (simply check the assignments for x, y in φ_r) but not minimal.

Minimal Models

For program Π and database D we define the set of minimal models:

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Example

$$\varphi_r = \forall x, y\,(HasEN(y) \wedge CoAuthor(x, y)) \rightarrow HasEN(x)$$

$$D = \{HasEN(\text{Erdos}), CoAuthor(\text{Erdos}, A1), CoAuthor(A3, A4)\}$$

Only $D \cup \{HasEN(A1)\}$ is a ***minimal*** model of φ_r!

Minimal Models

For program Π and database D we define the set of minimal models:

$$\textbf{MinMod}(\Pi, D) := \{\, D' \mid D \subseteq D', D' \text{ is a } \subseteq\text{-minimal model of } \Phi_\Pi \,\}$$

Theorem

For every Datalog program Π and database D:

$$|\,\textbf{MinMod}(\Pi, D)\,| = 1$$

A Proof Sketch

Let $H = D \cup \{\, R(\bar{c}) \mid R \text{ is an IDB}, \bar{c} \in Dom^{\#R} \,\}$. It is easy to check that $H \models \Phi_\Pi$: every possible right-hand side of an implication is in H. Hence $|\textbf{MinMod}(\Pi, D)| \geq 1$.

Assume $|\textbf{MinMod}(\Pi, D)| = \ell > 1$ and let $D_1, \ldots, D_\ell$ be the minimal models.
We show towards a contradiction that $D' = D_1 \cap \cdots \cap D_\ell$ is also a model of Φ_Π and $D' \supseteq D$:

1. For every $1 \leq i \leq \ell$, we have $D \subseteq D_i$. Hence $D \subseteq D'$.

2. For every $\varphi_r \in \Phi_\Pi$, if the left-hand side of an implication (for some assignment to the universally quantified variables) is in some D_i, then the right-hand side will also be in D_i. Hence, if the left-hand side of an implication is in D', so is the right-hand side.

Minimal Model Semantics

We can use this to define the result of a Datalog program Π on database D in an alternative way:

$$\Pi_{\text{mm}}(D) := \text{ the } \subseteq\text{-minimal model of } \Phi_D \text{ that contains } D$$

The definition of semantics of a Datalog query $q = (\Pi, R)$ remains unchanged, except for a change in what we consider the result of a program:

$$q(D) := \{(a_1, \ldots, a_{\#R}) \in Dom^{\#R} \mid R(a_1, \ldots, a_{\#R}) \in \Pi_{\text{mm}}(D)\}$$

Connecting the Two Semantics

Theorem

For every Datalog program Π and database D

$$\Pi_{\text{fp}}(D) = \Pi_{\text{mm}}(D).$$

Why?

As long as $\forall z_1, z_2, \ldots z_m \,.\, (L_1(\bar{x_1}) \wedge \cdots L_n(\bar{x_n})) \rightarrow R(\bar{x})$ is not satisfied, the immediate consequence operator will add the respective atom $R(\bar{c})$ for which the left-hand side of the implication was true. Hence $\Pi_{\text{fp}}(D)$ is a model of Φ_Π. Furthermore, by definition $D \subseteq \Pi_{\text{fp}}(D)$.

The immediate consequence operator cannot add any other atoms (hence, minimality).

Complexity of Datalog

Data/Combined Complexity

Combined Complexity:

Datalog-Eval is the problem, given input Datalog query $q = (\Pi, R)$ and database D, whether $q(D) \neq \emptyset$

Data Complexity:

In data complexity, we are interested in the complexity for fixed query, that is, only the database is the input.

Formally: for fixed Datalog query $q = (\Pi, R)$, q-Datalog-Eval is the problem, given database D, whether $q(D) \neq \emptyset$.

$\mathscr{C}$-hardness in data complexity means $\mathscr{C}$-hardness of q-Datalog-Eval for some q.

We say Datalog-Eval is in $\mathscr{C}$ in data complexity if q-Datalog-Eval is in $\mathscr{C}$ for every q,

A Simple Algorithm

From our discussion of semantics above we have $\bar{a} \in q(D) \iff R(\bar{a}) \in T_\Pi^\infty(D)$.

Remember from before that $\Pi_{\text{fp}}(D) = T_\Pi^n(D)$ for some $n \leq r \cdot |Dom(\Pi, D)|^a$.

Computing T_Π^i from T_Π^{i-1} requires for each rule r:

$$O(Dom(\Pi, D)^{vars(r)} \cdot |T_\Pi^{i-1}(D)|) \text{ time}$$

Number of functions from variables to domain.

Time to check if the function is a homomorphism into $T_\Pi^{i-1}(D)$

Recall that $|T_\Pi^{i-1}(D)|$ is at most $r \cdot |Dom(\Pi, D)|^a$!

A Simple Algorithm

From our discussion of semantics above we have $\bar{a} \in q(D) \iff R(\bar{a}) \in T_{\Pi}^{\infty}(D)$.

Remember from before that $\Pi_{\mathsf{fp}}(D) = T_{\Pi}^{n}(D)$ for some $n \leq r \cdot |Dom(\Pi, D)|^{a}$.

Computing T_{Π}^{i} from T_{Π}^{i-1} requires:

$$O(|\Pi| \cdot Dom(\Pi, D)^{maxvars} \cdot r \cdot |Dom(\Pi, D)|^{a}) \text{ time}$$

Repeating at most n times we have an algorithm to decide $\bar{a} \in q(D)$ in time

$$O(|\Pi| \cdot Dom(\Pi, D)^{maxvars} \cdot r^{2} \cdot |Dom(\Pi, D)|^{2a})$$

A Simple Algorithm

$$O(|\Pi| \cdot Dom(\Pi, D)^{maxvars} \cdot r^2 \cdot |Dom(\Pi, D)|^{2a})$$

Combined Complexity

The inputs are Π and D.

Both $maxvars$ and a depend on Π, i.e., our algorithm requires exponential time in combined complexity.

Data Complexity

The program Π is fixed, only the database D is an input.

All the exponents in our time bound are thus fixed, i.e., they are constants.

Our algorithm requires polynomial time in data complexity!

This algorithm is very basic.
Can we do significantly better?

Complexity of Datalog

Theorem

Datalog-Eval is ExpTime-complete.

Datalog-Eval is PTime-complete in data-complexity

Combined Complexity

We show ExpTime-hardness by reducing from acceptance in an exponential time Turing machine.

Let $L \in$ ExpTime, and let $M = (Q, \Sigma, \delta, q_0, q_\mathsf{T})$ be a deterministic TM that decides L in exponential time.

Reminder TM:
Q.. states
Σ...alphabet
δ ...transition function
q_0...start state
a ...accept state

Our goal: On input string w, compute in polynomial time a database D and Datalog query $q = (\Pi, \textbf{Accept})$ such that

$$M \text{ accepts } w \quad \Longleftrightarrow \quad q(D) \neq \emptyset$$

Combined Complexity

High-level overview of the reduction:

Any transition $\delta(q, a) = (q', b, dir)$ can be understood as a rule:

if at time t in state q the head reads a,
then at time $t + 1$, the state is q', the symbol a under the head gets
replaced by b, and the head is moved according to dir

We can express these statements easily in Datalog. If we can follow these computations for 2^m steps (where m is $|w|^k$ for some $k \in \mathbb{N}$), we can capture the full computation of M on w.

Constructing a Long Chain

For each $i \in [m-1]$:

$$
\begin{aligned}
Succ^{i+1}(z, \bar{x}, z, \bar{y}) &:- Succ^{i}(\bar{x}, \bar{y}), Low^{1}(z) \\
Succ^{i+1}(z, \bar{x}, z, \bar{y}) &:- Succ^{i}(\bar{x}, \bar{y}), High^{1}(z) \\
Succ^{i+1}(z, \bar{x}, v, \bar{y}) &:- Succ^{1}(z, v), Low^{i}(x), High^{i}(y) \\
High^{i+1}(x, \bar{y}) &:- High^{1}(x), High^{i}(\bar{y}) \\
Low^{i+1}(x, \bar{y}) &:- Low^{1}(x), Low^{i}(\bar{y}) \\
\preceq^{m}(\bar{x}, \bar{y}) &:- Succ^{m}(\bar{x}, \bar{y}) \\
\preceq^{m}(\bar{x}, \bar{z}) &:- \preceq^{m}(\bar{x}, \bar{y}), Succ^{m}(\bar{y}, \bar{z})
\end{aligned}
$$

Intuitively, this is a compact way to construct a sequence of length 2^m.

Example

$$Succ^{i+1}(z, \bar{x}, z, \bar{y}) :- \quad Succ^{i}(\bar{x}, \bar{y}), Low^{1}(z)$$
$$Succ^{i+1}(z, \bar{x}, z, \bar{y}) :- \quad Succ^{i}(\bar{x}, \bar{y}), High^{1}(z)$$
$$Succ^{i+1}(z, \bar{x}, v, \bar{y}) :- \quad Succ^{1}(z, v), High^{i}(\bar{x}), Low^{i}(\bar{y})$$
$$High^{i+1}(x, \bar{y}) :- \quad High^{1}(x), High^{i}(\bar{y})$$
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$$\preceq^{m} (\bar{x}, \bar{y}) :- \quad Succ^{m}(\bar{x}, \bar{y})$$
$$\preceq^{m} (\bar{x}, \bar{z}) :- \quad \preceq^{m} (\bar{x}, \bar{y}), Succ^{m}(\bar{y}, \bar{z})$$

$Succ^{1}(0,1), High(1), Low(0)$

$Succ^{2}(0,0,0,1)$

$Succ^{2}(1,0,1,1)$

$Succ^{2}(0,1,1,0)$

$High^{2}(1,1)$ $Low^{2}(0,0)$

We get the ordering
00 < 01 < 10 < 11

Example

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\end{aligned}
$$

$Succ^{1}(0,1), High(1)$ $Low(0)$

$Succ^{2}(0,0,0,1)$

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$Succ^{1}(0,1), High(1), Low(0)$

$Succ^{2}(0,0,0,1)$

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$Succ^{2}(0,1,1,0)$

We get the ordering
00 < 01 < 10 < 11

$High^{2}(1,1)$ $Low^{2}(0,0)$

$Succ^{2}(0,0,1,0,1,0)$

⋮

$Succ^{2}(0,0,0,0,0,1)$

⋮

The Database

We only need a very basic database from which our successor relationship can be constructed.

$$D = \{\ Succ^1(0,1),\ High^1(1),\ Low^1(0)\ \}$$

Note that the database in this construction is *independent* of the input w!
Hence this reduction does not work to establish the complexity in data complexity.

The Starting State

For input word $w = a_0 a_1 \cdots a_\ell$ we add the following rules:

$$State_s(\bar{x}) :- \quad Low^m(\bar{x})$$ "At time 0, the machine is in the starting state"

$$Symbol_{a_0}(\bar{x}, \bar{x}) :- \quad Low^m(\bar{x})$$ "At time 0, the symbol a_0 is on cell 0 of the tape"

$$Symbol_{a_1}(\bar{x}_0, \bar{x}_1) :- \quad Low^m(\bar{x}_0), Succ^m(\bar{x}_0, \bar{x}_1)$$ "At time 0, the symbol a_1 is on cell 1 of the tape"

$$\vdots$$

$$Symbol_{a_\ell}(\bar{x}_0, \bar{x}_\ell) :- \quad Low^m(\bar{x}_0), Succ^m(\bar{x}_0, \bar{x}_1), \ldots Succ^m(\bar{x}_{\ell-1}, \bar{x}_\ell)$$

$$Symbol_{\sqcup}(\bar{x}_0, \bar{y}) :- \quad Low^m(\bar{x}_0), Succ^m(\bar{x}_0, \bar{x}_1), \ldots Succ^m(\bar{x}_{\ell-1}, \bar{x}_\ell), \preceq^m (\bar{x}_\ell, \bar{y})$$

$$Head(\bar{x}, \bar{x}) :- \quad Low^m(\bar{x})$$

We use $Symbol_a(t, c)$ to express that at time t cell c contains symbol a.

Similarly, $Head(t, c)$ means that at time t, the head is over cell c.

Transitions as Rules

For every transition $\delta(q, a) = (q', b, \rightarrow)$ add the rules:

$$
\begin{aligned}
State_{q'}(\bar{z}) :- \quad & State_q(\bar{x}), Head(\bar{x}, \bar{y}), Symbol_a(\bar{x}, \bar{y}), Succ^m(\bar{x}, \bar{z}) \\
Symbol_b(\bar{z}, \bar{y}) :- \quad & State_q(\bar{x}), Head(\bar{x}, \bar{y}), Symbol_a(\bar{x}, \bar{y}), Succ^m(\bar{x}, \bar{z}) \\
Head(\bar{z}, \bar{v}) :- \quad & State_q(\bar{x}), Head(\bar{x}, \bar{y}), Symbol_a(\bar{x}, \bar{y}), Succ^m(\bar{x}, \bar{z}), Succ^m(\bar{y}, \bar{v})
\end{aligned}
$$

It is straightforward to adapt the $Head$ rule to the other directions.

Inertia and Acceptance

The final missing part is to preserve unchanged symbols over time:

$$Symbol_a(\bar{v}, \bar{y}) :- \quad Symbol_a(\bar{x}, \bar{y}),\ Head(\bar{x}, \bar{z}),\ \preceq^m (\bar{y}, \bar{z}),\ Succ^m(\bar{x}, \bar{v})$$
$$Symbol_a(\bar{v}, \bar{y}) :- \quad Symbol_a(\bar{x}, \bar{y}),\ Head(\bar{x}, \bar{z}),\ \preceq^m (\bar{z}, \bar{y}),\ Succ^m(\bar{x}, \bar{v})$$

The first rule propagates the cells that come before the head to the next timepoint, the second propagates the cells after the head.

Finally, we check whether our simulation of the machine reaches the accepting state q_T

$$\mathsf{Accept} :- \quad State_{q_\mathsf{T}}(\bar{x})$$

Data Complexity

For data complexity we need to reduce some problem to answering a fixed Datalog query.

That is, we give a fixed query $q = (\Pi, R)$ that doesn't depend on the problem input. For every input w, the reduction computes a database D such that

$$q(D) \neq \emptyset \iff w \text{ is an accepting input}$$

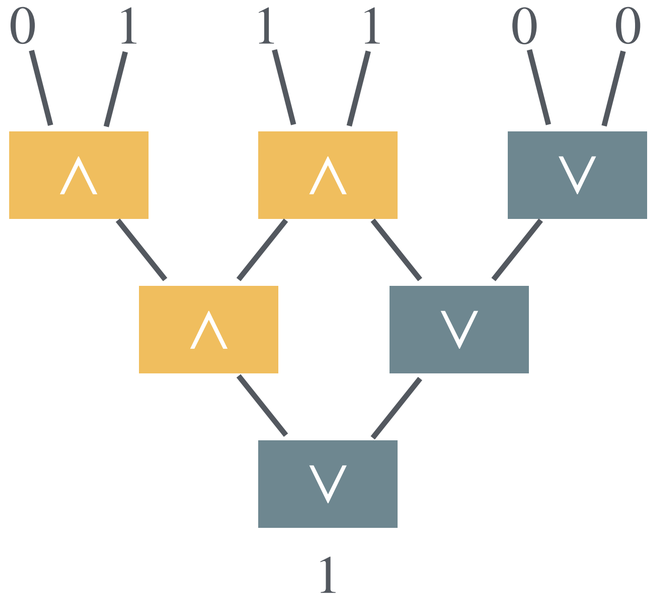
Data Complexity

We will reduce from monotone Boolean-circuit evaluation (MCVP)

MCVP

Input: An n-input, monotone Boolean circuit C, and a string $w \in \{0,1\}^n$

Output: accepts iff C outputs true on input w.



Data Complexity

We fix the Datalog query $q = (\Pi, \textsf{Accept})$ with program Π:

$$
\begin{aligned}
True(x) &:- Or(x, y, z), True(y) \\
True(x) &:- Or(x, y, z), True(z) \\
True(x) &:- And(x, y, z), True(y), True(z) \\
\textsf{Accept} &:- End(x), True(x)
\end{aligned}
$$

The program simulates a monotone Boolean circuit specified in the database.

To complete the proof we need to give a *LogSpace* algorithm that takes a circuit as input and outputs a corresponding database.

Stratified Negation

Negation in Datalog

$$P \ :- \neg Q$$
$$Q \ :- \neg P$$

What is supposed to happen here?

Key problem:
Relation is defined on the basis of its own negation

Semipositive Programs

A Datalog program with negation is semipositive if only EDB relation names are negated.

Say we have EDB relation name $Knows$, then the following program is semipositive:

$$Stranger(x, y) \ :- \neg Knows(x, y)$$

Semantics of semipositive programs is straightforward:

1. For every negated EDB introduce relation name $R_\neg$ and add $R_\neg^D = Dom^{\#R} \backslash R^D$ to the database
2. replace every occurrence of $\neg R(\bar{x})$ with with $R_\neg(\bar{x})$
3. The program now has no more negation, evaluate as usual.

Stratified Negation

Semipositive is too restrictive, but often programs can be seen as a sequence of multiple semipositive programs!

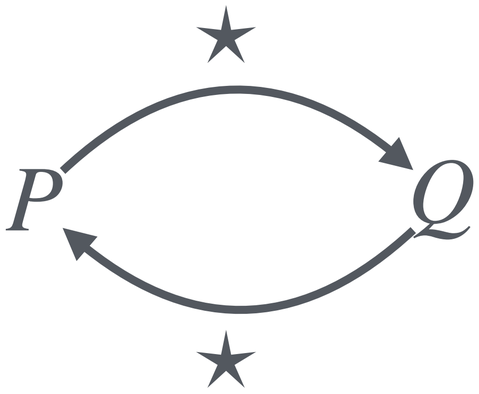
Dependency Graph of program Π is a directed graph:

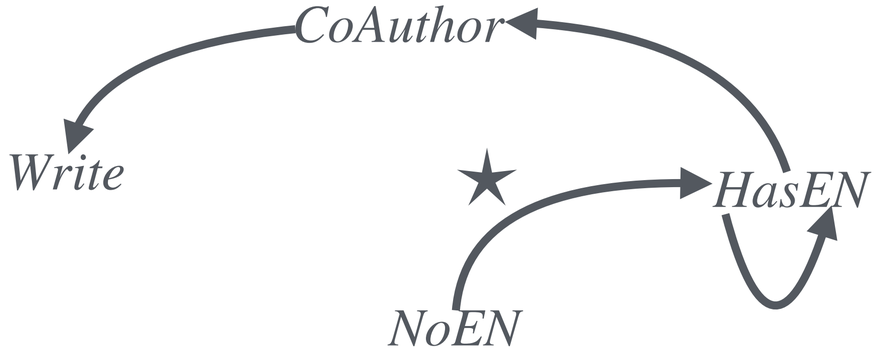
- Vertices $:=$ relation names in Π
- There is an arc $p \to q$ if there is a rule in Π where p is the head relation symbol, and q is in the body.
- We mark arcs $p \to q$ with a star $\star$ if q in the body is negated.

Dependency Graph

$$P \text{ :- } \neg Q$$
$$Q \text{ :- } \neg P$$

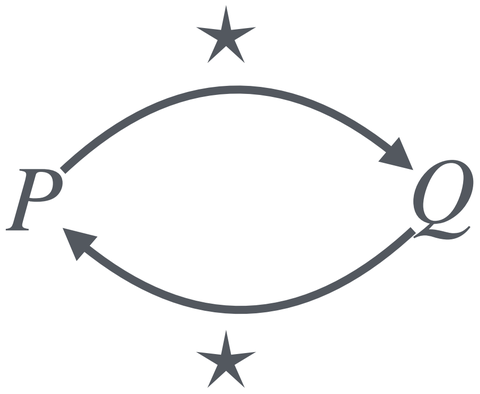
$$\begin{aligned} CoAuthor(y, x) &\text{ :- } Write(x, p),\ Write(y, p) \\ HasEN(x) &\text{ :- } HasEN(y),\ CoAuthor(y, x) \\ NoEN(x) &\text{ :- } \neg HasEN(x) \\ HasEN(\text{Erdos}) &. \end{aligned}$$





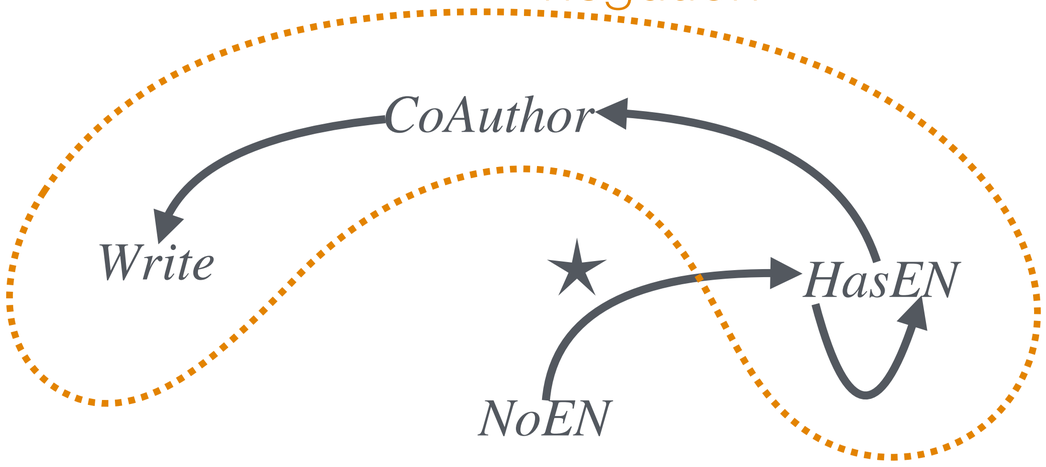
Stratifiable Programs

If the dependency graph of Π has no directed cycle that contains an edge marked with $\star$ we call Π stratifiable.



Not stratifiable

Intuition: we can solve this “stratum" before checking negation



Stratifiable

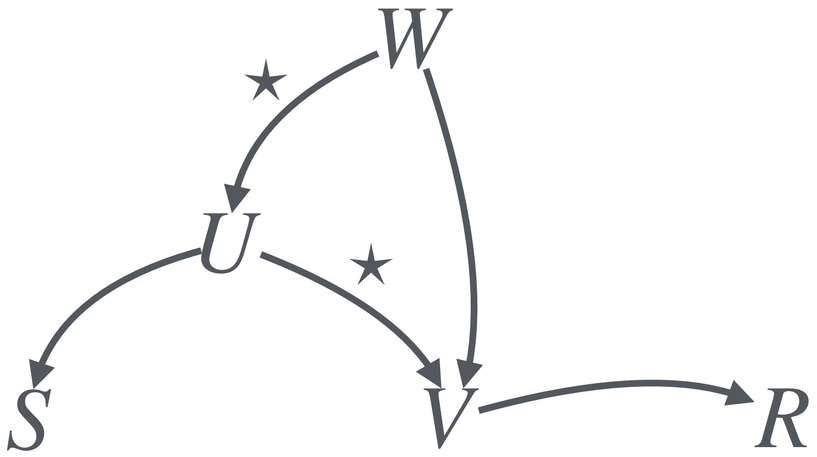
Stratification

A stratification is a function $\lambda : \mathbf{Rel} \to \mathbb{N}$ that assign to every relation name the a number such that:

- If $R \in \mathbf{Rel}$ is an EDB, then $\sigma(R) = 0$.
- If there is a rule with R in its head and P in its body, then $\sigma(R) \geq \sigma(P)$.
- If there is a rule with R in its head and $\neg P$ in its body, then $\sigma(R) > \sigma(P)$

Example

$$
\begin{aligned}
V(x,y) &:- R(x,x), R(y,y) \\
U(x,y) &:- S(x,y), S(y,z), \neg V(x,y) \\
W(x,y) &:- \neg U(x,y), V(y,x)
\end{aligned}
$$



Level 3: W

Level 2: U

Level 1: R, S, V

Stratified Negation — Semantics

For a given stratification σ, partition the program Π into programs $\Pi_1, \Pi_2, \ldots$:
Rule r with head relation name R is in program $\Pi_{\sigma(R)}$.

$$
\begin{array}{lrcl}
\Pi_1 & V(x,y) & :- & R(x,x), R(y,y) \\
\Pi_2 & U(x,y) & :- & S(x,y), S(y,z), \neg V(x,y) \\
\Pi_3 & W(x,y) & :- & \neg U(x,y), V(y,x)
\end{array}
$$

Define $D_0 = D$, $D_i = \Pi_i(D_{i-1})$. The result of a stratified program $\Pi_\sigma(D) := D_\ell$ where ℓ is the maximum level assigned by σ.

Note that this way, every program Π_i is semipositive as every negated atom occurs in a lower stratum $\Rightarrow$ it becomes an EDB in this stratum-wise execution.

Stratified Negation — Semantics

One problem is left, semantics depend on a specific stratification σ.
However, actually any stratification leads to the same result, providing us with robust semantics for Datalog with stratified negation.

Theorem

Let σ_1, σ_2 be two stratifications of Datalog program Π. Then $\Pi_{\sigma_1}(D) = \Pi_{\sigma_2}(D)$.

Beyond Datalog

The Real World

Modern Datalog engine that additionally supports stratified aggregation, value invention, and many QoL extensions.

https://knowsys.github.io/nemo-doc/

Try it in the browser:
https://tools.iccl.inf.tu-dresden.de/nemo/

Datalog-based language that compiles to SQL.

Specialised for data analysis.

https://logica.dev/

Negation

Well-founded Negation

Three-values logic where atoms can also have “undefined” truth.

For example, in mutual recursion, both atoms have undefined truth.

Stable Negation

Consider anything that is not necessarily true as false. Programs can have multiple incomparable models!

$$P :- \neg Q$$
$$Q :- \neg P$$

has two stable models:

$\{P\}$ and $\{Q\}$

Datalog$^{\pm}$

We commonly need to deal with incomplete data.
We might need to invent new values!

$$\exists x\ Manager(x, y)\ :-\ Employee(y)$$

Creates a much more powerful language (the $+$).
So powerful answering queries becomes undecidable.

We therefore study restrictions that allow for value invention but keep the language decidable (the $-$).

Restricted Datalog

Restrictions to standard Datalog have also been studied.
Some examples:

- ***Linear Datalog***
  All rules have at most one IDB in their body. This guarantees that any recursion is “linear”, i.e., there are no joins on recursion.
  PSpace-complete in combined complexity, NL-complete in data complexity.

- ***Non-recursive Datalog***
  No recursion is allowed, i.e., no cycles in the dependency graph at all.
  We will see in the next lecture that non-recursive Datalog with negation this is equivalent to relational algebra.