Database Theory

Conjunctive Queries

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Motivation

We've seen that we are limited in many things we want to do when it comes to powerful languages like FO/RA.

Let us instead study a restricted subclass of queries that lies at the core of important data retrieval tasks.

Conjunctive Queries

(Can also contain constants.)

$$\{\, \bar{y} \mid \exists \bar{z}\, R_1(\bar{x}_1) \wedge R_2(\bar{x}_2) \wedge \cdots \wedge R_k(\bar{x}_k) \,\}$$

We call queries of this form Conjunctive Queries (CQs).

That is, conjunctive queries are FO queries using only the connectives $\exists$ and $\wedge$.

Conjunctive Queries

$$\{\, \bar{y} \mid \exists \bar{z}\, R_1(\bar{x}_1) \wedge R_2(\bar{x}_2) \wedge \cdots \wedge R_k(\bar{x}_k) \,\}$$

$$\pi_{\bar{y}} \left(R_1 \bowtie R_2 \bowtie \cdots \bowtie R_k \right)$$

(Assuming attributes for R_i are $\bar{x}_i$, otherwise simply rename)

Conjunctive Queries

$$\{\, \bar{y} \mid \exists \bar{z}\, R_1(\bar{x}_1) \wedge R_2(\bar{x}_2) \wedge \cdots \wedge R_k(\bar{x}_k) \,\}$$

$$\pi_{\bar{y}} \left(R_1 \bowtie R_2 \bowtie \cdots \bowtie R_k \right)$$

Conjunctive queries correspond so-called join queries in RA.
That is, RA queries that only use projection, renaming, selection, and joins.

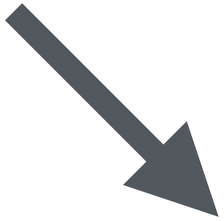
Conjunctive Queries

Recall our SQL example in the lecture on the relational model.

```
SELECT course.name, student.name
FROM course, student, enrolled
WHERE course.name = enrolled.course AND
      course.sem = enrolled.sem AND
      student.id = enrolled.student AND
      enrolled.sem = 'WS24';
```

Conjunctive Queries

```sql
SELECT course.name, student.name
FROM course, student, enrolled
WHERE course.name = enrolled.course AND
      course.sem = enrolled.sem AND
      student.id = enrolled.student AND
      enrolled.sem = 'WS24';
```

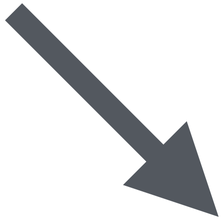


$$\{(c, s) \mid \exists sid, l, dob, active \,.\, Enrolled(c, \text{WS24}, sid) \wedge Course(c, \text{WS24}, l) \wedge Student(sid, s, dob, active)\}$$

Conjunctive Queries

```
SELECT course.name, student.name
FROM course, student, enrolled
WHERE course.name = enrolled.course AND
      course.sem = enrolled.sem AND
      student.id = enrolled.student AND
      enrolled.sem = 'WS24';
```

CQs cover the core part of most SQL queries!



$$\{(c, s) \mid \exists sid, l, dob, active \,.\, Enrolled(c, \text{WS24}, sid) \wedge Course(c, \text{WS24}, l) \wedge Student(sid, s, dob, active)\}$$

Conjunctive Queries

- Conjunctive queries form the key part of most data retrieval tasks.

- Join queries
- Datalog rule bodies are CQs
- Basis for many other query languages, e.g., Conjunctive Regular Path Queries.

Conjunctive Queries

- Conjunctive queries form the key part of most data retrieval tasks.
- Optimising CQs can help to optimise the most expensive part of practical join evaluations

```
select min(ps_supplycost)
from
      partsupp,
      supplier,
      nation,
      region
where
      p_partkey = ps_partkey
      and s_suppkey = ps_suppkey
      and s_nationkey = n_nationkey
      and n_regionkey = r_regionkey
      and r_name = 'Asia'
```

Real systems will evaluate a CQ first and then evaluate the min aggregate on the result of the CQ.

Conjunctive Queries

- Conjunctive queries form the key part of most data retrieval tasks.
- Optimising CQs can help to optimise the most expensive part of practical join evaluations
- Complexity for results for CQs also give us lower bounds for more complex queries that often have CQs at their core.

```
select min(ps_supplycost)
from
      partsupp,
      supplier,
      nation,
      region
where
      p_partkey = ps_partkey
      and s_suppkey = ps_suppkey
      and s_nationkey = n_nationkey
      and n_regionkey = r_regionkey
      and r_name = 'Asia'
```

The query intuitively will be at least as hard to solve as the underlying CQ (without the min aggregate).

Equivalence & Containment

Query Containment

- For queries q_1, q_2 we say that q_1 is contained in q_2 (in symbols, $q_1 \subseteq q_2$) if $q_1(D) \subseteq q_2(D)$ for every database D.

- Equivalence of two queries q_1, q_2 (in symbols, $q_1 \equiv q_2$) is defined as

$$q_1 \equiv q_2 \iff q_1 \subseteq q_2 \text{ and } q_2 \subseteq q_1$$

See the Trakthenbrot exercise sheet!

Given q_1, q_2 in RA, there is no algorithm to decide $q_1 \subseteq q_2$.

But if q_1, q_2 are CQs then the problem is decidable!

Query Containment Example

Consider the following two queries

$$q_1 := \{(x, y) \mid \exists z\, R(y, x) \wedge R(x, z)\}$$

$$q_2 := \{(x, y) \mid R(y, x) \wedge R(x, y)\}$$

R

A	B
1	2
3	3
2	3

Intuitively it seems clear that q_1 describes a weaker requirement: x only needs to reach some z, whereas it needs to reach specifically y in q_2.

$$q_1(D) = \{\ (2,1), (3,2), (3,3)\}$$
$$q_2(D) = \{\ (3,3)\}$$

We would therefore suspect that $q_2(D) \subseteq q_1(D)$ for all D. But how to prove it?

The Tableau of a CQ

Basic Idea: we can represent a CQ as a database.

The tableau $\mathsf{Tbl}(q)$ of a CQ q is the database where the tuples of relation R are all of the term lists that occur for R in the query (+ a relation for the output variables).

Consider the following query:

$$\{\ (x, y) \mid \exists wz\ B(x, y) \wedge R(y, z) \wedge R(y, w) \wedge R(w, y)\ \}$$

We write $\mathsf{Tbl}^*(q)$ for the tables without the special **Out** relation for output variables.

Out

1	*2*
x	y

B

1	*2*
x	y

R

1	*2*
y	z
y	w
w	y

Homomorphisms

A homomorphism of two databases D_1, D_2 is a function $h : Dom(D_1) \to Dom(D_2)$ such that:

$$(c_1, \ldots, c_n) \in R^{D_1} \implies (h(c_1), \ldots, h(c_n)) \in R^{D_2} \qquad \forall R \in \mathsf{Rel}$$

Example

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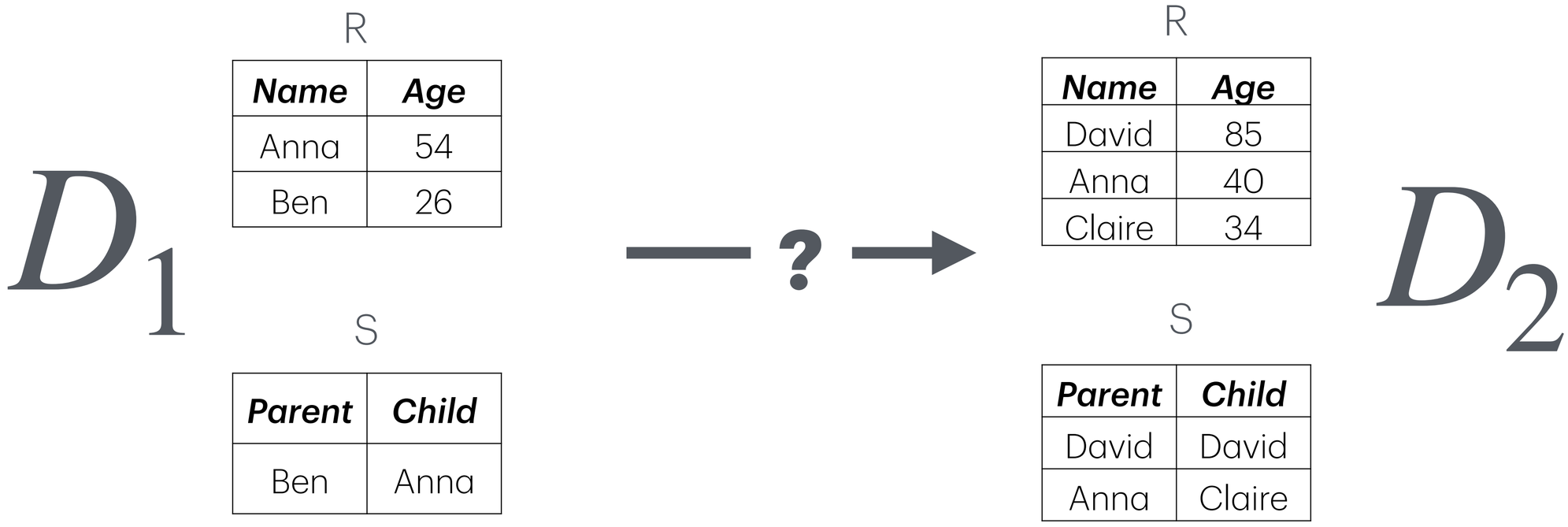

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D_1

R

Name	***Age***
Anna	54
Ben	26

S

Parent	***Child***
Ben	Anna

$$Anna \mapsto Claire$$
$$Ben \mapsto Anna$$
$$54 \mapsto 34$$
$$26 \mapsto 40$$

May seem weird in meaning but the "structure" is preserved!

R

Name	***Age***
David	85
Anna	40
Claire	34

S

Parent	***Child***
David	David
Anna	Claire

D_2

Example

A homomorphism of two databases D_1, D_2 is a function $h : Dom(D_1) \to Dom(D_2)$ such that:

$$(c_1, \ldots, c_n) \in R^{D_1} \implies (h(c_1), \ldots, h(c_n)) \in R^{D_2} \qquad \forall R \in \mathbf{Rel}$$

D_1

R

Name	***Age***
Anna	54
Ben	26

S

Parent	***Child***
Ben	Anna

$$Anna \mapsto David$$
$$Ben \mapsto David$$
$$54 \mapsto 85$$
$$26 \mapsto 85$$

Nothing says the function must be injective

R

Name	***Age***
David	85
Anna	40
Claire	34

S

Parent	***Child***
David	David
Anna	Claire

D_2

Example

A homomorphism of two databases D_1, D_2 is a function $h : Dom(D_1) \to Dom(D_2)$ such that:

$$(c_1, \ldots, c_n) \in R^{D_1} \implies (h(c_1), \ldots, h(c_n)) \in R^{D_2} \qquad \forall R \in \mathsf{Rel}$$

D_1

R

Name	***Age***
Anna	54
Ben	26

S

Parent	***Child***
Ben	Anna

Not a hom!

$Anna \mapsto Claire$

$Ben \mapsto Claire$

$54 \mapsto 34$

$26 \mapsto 34$

But that doesn't always work

R

Name	***Age***
David	85
Anna	40
Claire	34

S

Parent	***Child***
David	David
Anna	Claire

D_2

Homomorphism Theorem

Theorem

Let q_1, q_2 be CQs. Then

$$q_1 \subseteq q_2 \quad \text{if and only if} \quad \mathsf{Tbl}(q_2) \xrightarrow{hom} \mathsf{Tbl}(q_1)$$

As a result we have an "easy" algorithm for deciding containment for CQs:

1. Compute $\mathsf{Tbl}(\cdot)$ for both queries (trivial).
2. Check if there is a homomorphism (in NP).

Careful!
Only holds for
set semantics!

Homomorphism Theorem

Consider the following two queries

$$q_1 := \{(x, y) \mid \exists z\, R(y, x) \wedge R(x, z)\}$$

$$q_2 := \{(x, y) \mid \exists wu\, R(y, x) \wedge R(w, x) \wedge R(x, u)\}$$

Tbl(q_1)

Out

1	2
x	y

R

1	2
y	x
x	z

Tbl(q_2)

Out

1	2
x	y

R

1	2
y	x
w	x
x	u

Homomorphism Theorem

Tbl(q_1)

Out

1	2
x	y

R

1	2
y	x
x	z

Tbl(q_2)

Out

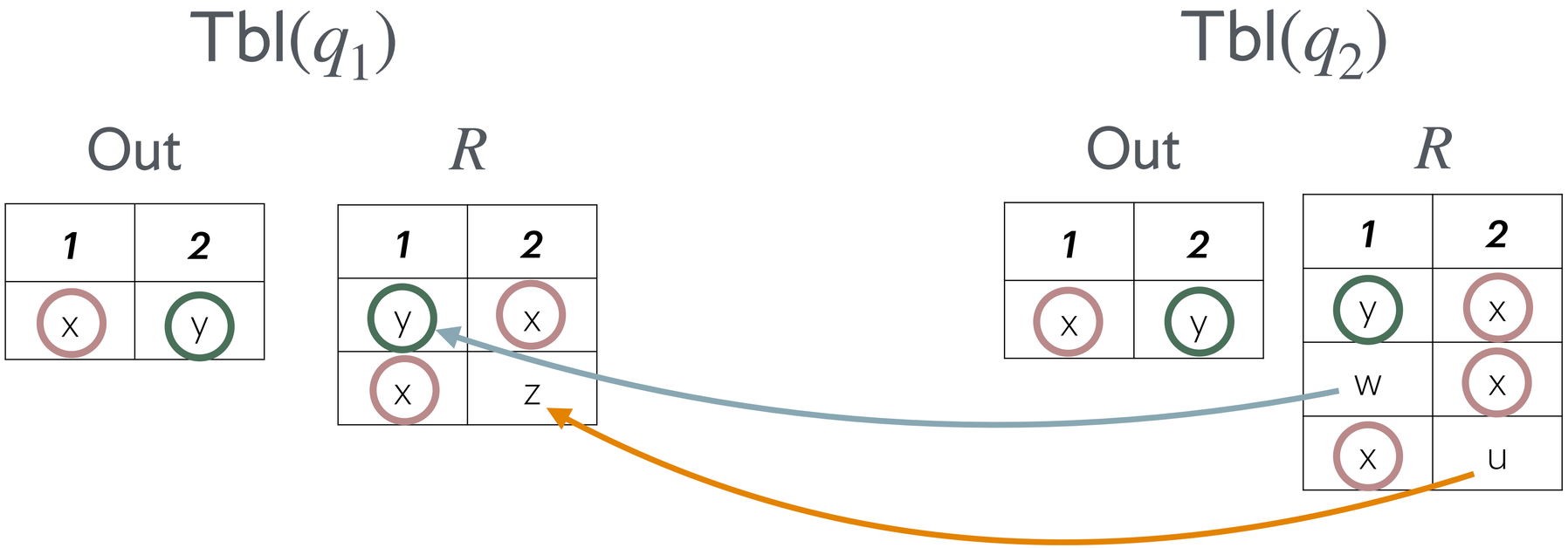
1	2
x	y

R

1	2
y	x
w	x
x	u

Since **Out** has only one tuple, a homomorphism h must necessarily have $h(x) = x$ and $h(y) = y$.

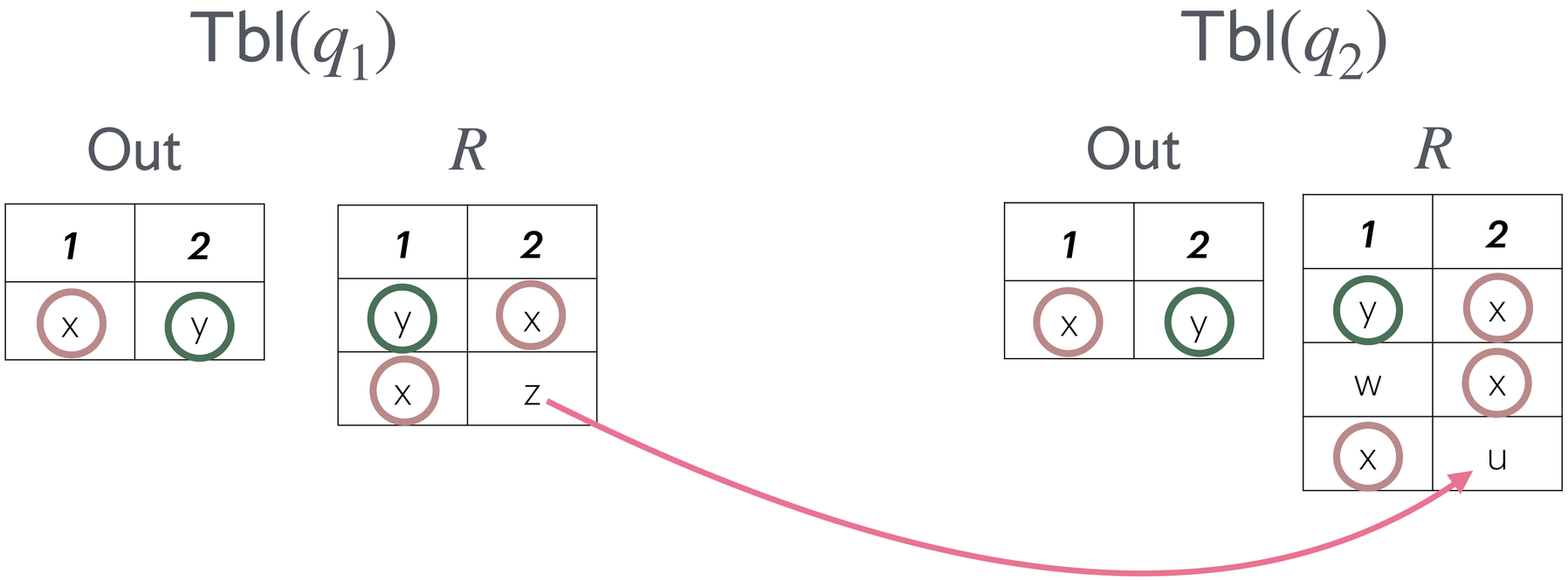
Homomorphism Theorem



Since **Out** has only one tuple, a homomorphism h must necessarily have $h(x) = x$ and $h(y) = y$.

With $h(w) = y$ and $h(u) = z$ we get a homomorphism $\mathsf{Tbl}(q_2) \xrightarrow{hom} \mathsf{Tbl}(q_1)$.

Homomorphism Theorem



Since **Out** has only one tuple, a homomorphism h must necessarily have $h(x) = x$ and $h(y) = y$.

With $h(z) = y$ and $h(u) = z$ we get a homomorphism $\mathsf{Tbl}(q_1) \xrightarrow{hom} \mathsf{Tbl}(q_2)$.

Homomorphism Theorem

Consider the following two queries

$$q_1 := \{(x, y) \mid \exists z\, R(y, x) \wedge R(x, z)\}$$

$$q_2 := \{(x, y) \mid \exists wu\; R(y, x) \wedge R(w, x) \wedge R(x, u)\}$$

We've seen that $\mathsf{Tbl}(q_1) \xrightarrow{hom} \mathsf{Tbl}(q_2)$ and $\mathsf{Tbl}(q_2) \xrightarrow{hom} \mathsf{Tbl}(q_1)$.

By the Homomorphism Theorem that means $q_1 \subseteq q_2$ and $q_2 \subseteq q_1$, i.e., $q_1 \equiv q_2$!

Why?

Important observation

Let q be a CQ with free variables $\bar{y}$. For any database D, we have $\bar{c} \in q(D)$ ***if and only if*** there is a homomorphism h from $\mathsf{Tbl}^*(q)$ to D such that $h(\bar{y}) = \bar{c}$.

$$q_2 := \quad \{(x, y) \mid \exists wu\ R(y, x) \wedge R(w, x) \wedge R(x, u)\}$$

$(a, b) \in q_2(D)$ means that there is an interpretation I such that

1. $R(I(y), I(x)) \in D$, $R(I(w), I(x)) \in D$, and $R(I(x), I(u)) \in D$.
2. and $I(x) = a$ and $I(y) = b$.

So I is precisely a homomorphism from $\mathsf{Tbl}^*(q_2)$ to D!

Proof Idea

If $q_1 \subseteq q_2$, then $\mathsf{Tbl}(q_2) \xrightarrow{hom} \mathsf{Tbl}(q_1)$

By assumption $q_1(D) \subseteq q_2(D)$. So take $\mathsf{Tbl}^*(q_1)$ as database D. Let $\bar{y}$ be the free variables of q_1. We have that $(x, y) \in q_1(\mathsf{Tbl}^*(q_1))$ since we can just map every variable to itself.

$\mathsf{Tbl}^*(q_1)$ R

1	2
y	x
x	z

$h(x) = x, h(y) = y, h(z) = z$

$\mathsf{Tbl}^*(q_1)$ R

1	2
y	x
x	z

So $(x, y) \in q_1(\mathsf{Tbl}^*(q_1))$

Example queries from before

$q_1 := \{(x, y) \mid \exists z\, R(y, x) \wedge R(x, z)\}$

$q_2 := \{(x, y) \mid \exists wu\, R(y, x) \wedge R(w, x) \wedge R(x, u)$

Proof Idea

If $q_1 \subseteq q_2$, then $\mathsf{Tbl}(q_2) \xrightarrow{hom} \mathsf{Tbl}(q_1)$

By assumption $q_1(D) \subseteq q_2(D)$. So take $\mathsf{Tbl}^*(q_1)$ as database D. Let $\bar{y}$ be the free variables of q_1. We have that $(x, y) \in q_1(\mathsf{Tbl}^*(q_1))$ since we can just map every variable to itself.

Then also $(x, y) \in q_2(\mathsf{Tbl}^*(q_1))$. By the key observation:

$$\bar{y} \in q_2(\mathsf{Tbl}^*(q_1)) \iff \mathsf{Tbl}^*(q_2) \xrightarrow{hom} \mathsf{Tbl}^*(q_1)$$

That is, the tuple in the **Out** relation of $\mathsf{Tbl}(q_2)$ maps into a tuple of **Out** of $\mathsf{Tbl}(q_1)$.
Furthermore, that homomorphism maps the free variables of q_2 to the free variables of q_1.

We then have $\mathsf{Tbl}(q_2) \xrightarrow{hom} \mathsf{Tbl}(q_1)$.

Proof Idea

Example queries from before

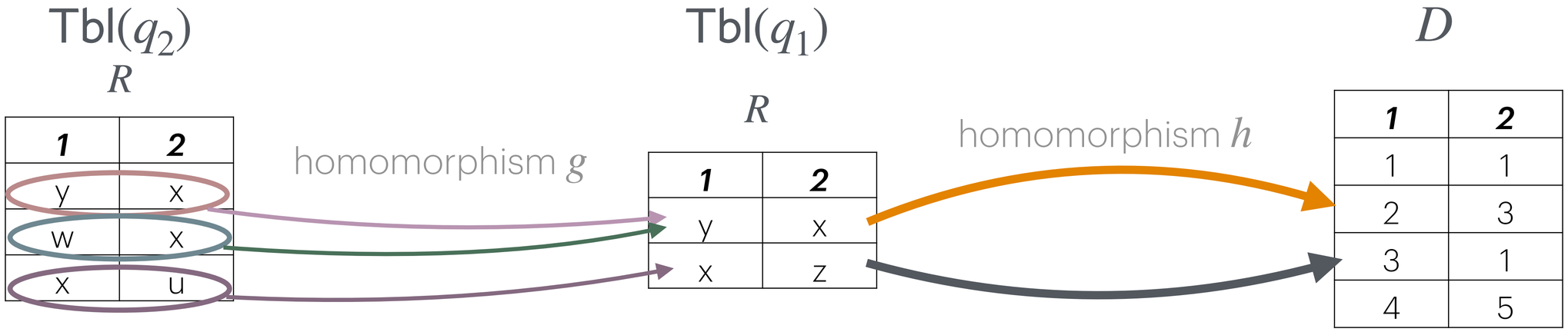
$q_1 := \{(x, y) \mid \exists z\, R(y, x) \wedge R(x, z)\}$

$q_2 := \{(x, y) \mid \exists wu\, R(y, x) \wedge R(w, x) \wedge R(x, u)\}$

If $\mathsf{Tbl}(q_2) \xrightarrow{hom} \mathsf{Tbl}(q_1)$, then $q_1 \subseteq q_2$

If $\bar{c}$ is an answer of q_1 on some database D then there is a homomorphism $h : \mathsf{Tbl}^*(q_1) \to D$ that maps the output variables of q_1 to $\bar{c}$.

Let g be the homomorphism from $\mathsf{Tbl}(q_2)$ to $\mathsf{Tbl}(q_1)$. It is not hard to see that $h \circ g$ is homomorphism from $\mathsf{Tbl}^*(q_2)$ to D that also maps the output of variables of q_2 to $\bar{c}$.

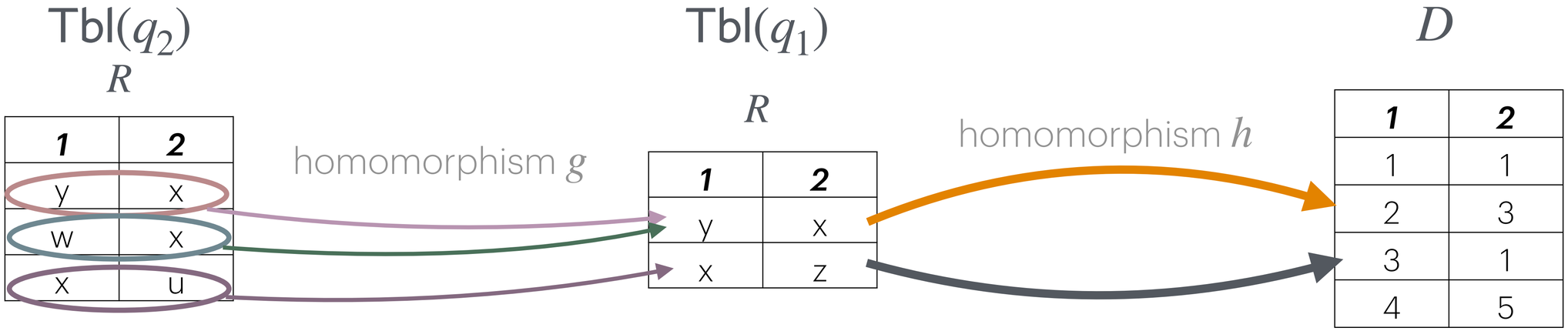


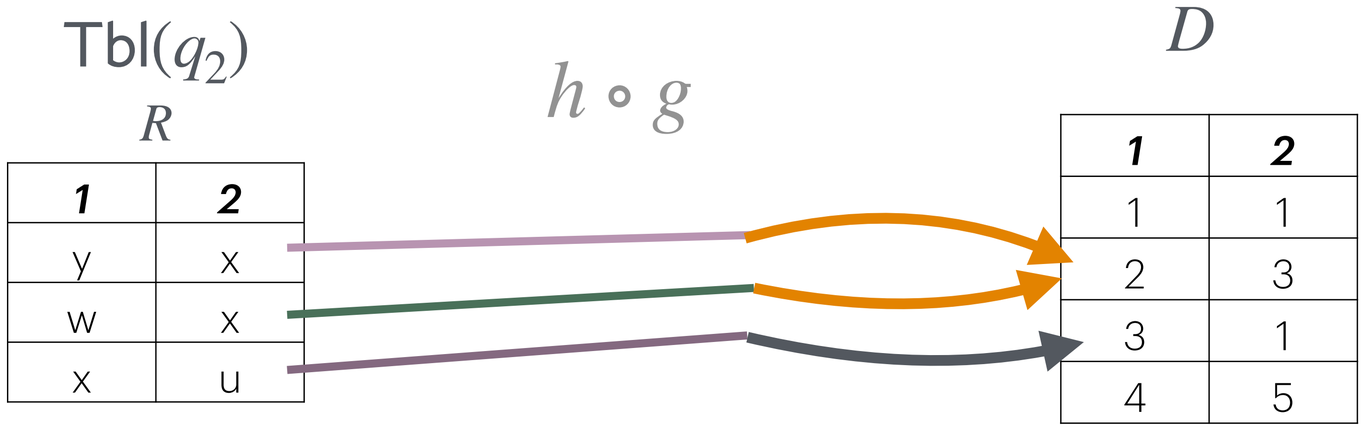
Proof Idea

Example queries from before

$q_1 := \{(x, y) \mid \exists z\, R(y, x) \wedge R(x, z)\}$

$q_2 := \{(x, y) \mid \exists wu\, R(y, x) \wedge R(w, x) \wedge R(x, u)\}$





Still a homomorphism!

Output variables map to the same values in D!

Proof Idea

Example queries from before

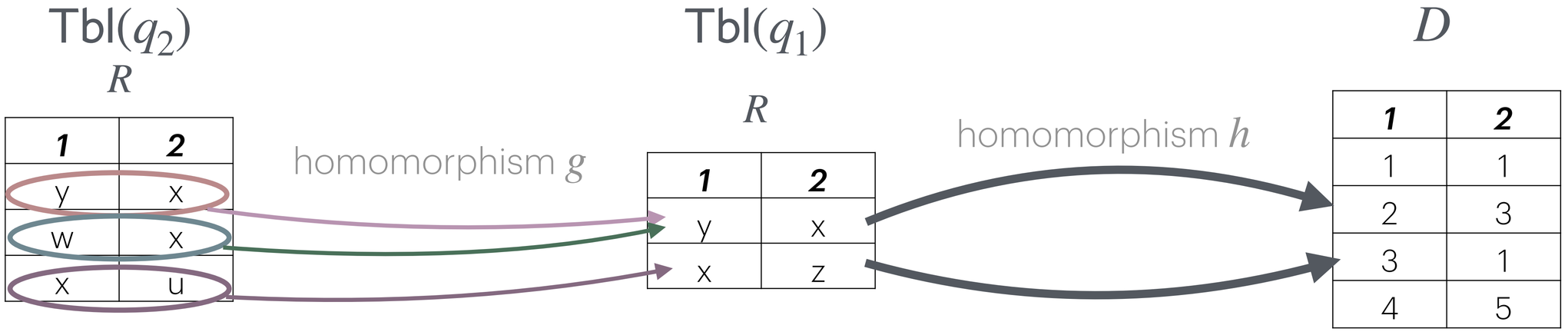
$q_1 := \{(x, y) \mid \exists z\, R(y, x) \wedge R(x, z)\}$

$q_2 := \{(x, y) \mid \exists wu\, R(y, x) \wedge R(w, x) \wedge R(x, u)\}$

If $\mathsf{Tbl}(q_2) \xrightarrow{hom} \mathsf{Tbl}(q_1)$, then $q_1 \subseteq q_2$

If $\bar{c}$ is an answer of q_1 on some database D, then there is a homomorphism $h : \mathsf{Tbl}^*(q_1) \to D$ that maps the output variables of q_1 to $\bar{c}$.

Let g be the homomorphism from $\mathsf{Tbl}(q_2)$ to $\mathsf{Tbl}(q_1)$. It is not hard to see that $h \circ g$ is homomorphism from $\mathsf{Tbl}^*(q_2)$ to D that also maps the output of variables of q_2 to $\bar{c}$.



Query Minimisation

Minimising?

Goal:

Given a CQ q, we want the equivalent CQ q' with the least amount of atoms.

Formally, a CQ q is minimal if there does not exist a CQ q' such that:

a) $q' \equiv q$

b) q' has fewer atoms (=terms in the conjunction) than q

We would like to replace a CQ with its minimal equivalent CQ before evaluating it.

How do we find this minimal equivalent CQ?

To minimise CQ q, it is enough to check only those queries obtained by deleting atoms from q!

Minimisation by Deletion

Assume CQ $q = \{\, \bar{y} \mid \exists \bar{z}\; R_1(\bar{x}_1) \wedge R_2(\bar{x}_2) \wedge \cdots \wedge R_k(\bar{x}_k) \,\}$.

Furthermore, assume q has an equivalent CQ

$q' = \{\, \bar{y}' \mid \exists \bar{z}'\; S_1(\bar{x}'_1) \wedge S_2(\bar{x}'_2) \wedge \cdots \wedge S_j(\bar{x}'_j) \,\}$ with $j < k$.

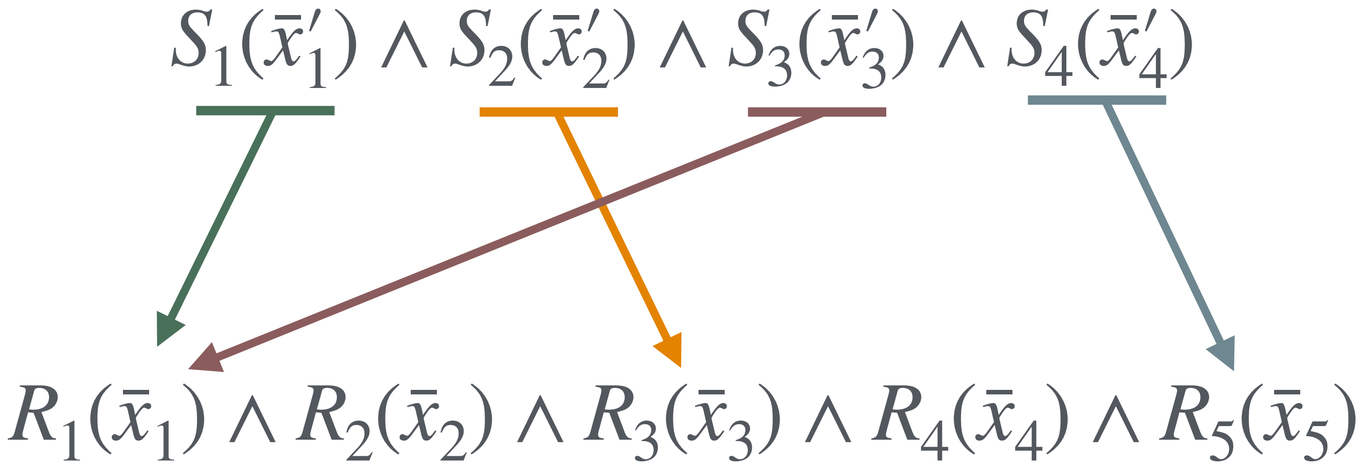
By the Homomorphism Theorem there are homomorphisms:

$$f : \mathsf{Tbl}(q) \to \mathsf{Tbl}(q') \quad \text{and} \quad g : \mathsf{Tbl}(q') \to \mathsf{Tbl}(q)$$

Minimisation by Deletion

We have that $g : \mathsf{Tbl}(q') \to \mathsf{Tbl}(q)$ maps every $S_\alpha(\bar{x}'_\alpha) \in \mathsf{Tbl}(q')$ into some $R_{i_\alpha}(\bar{x}_{i_\alpha}) \in \mathsf{Tbl}(q)$ with $S_\alpha = R_{i_\alpha}$ and $\bar{x}_{i_\alpha} = h(\bar{x}'_\alpha)$.

the mapping g



Minimisation by Deletion

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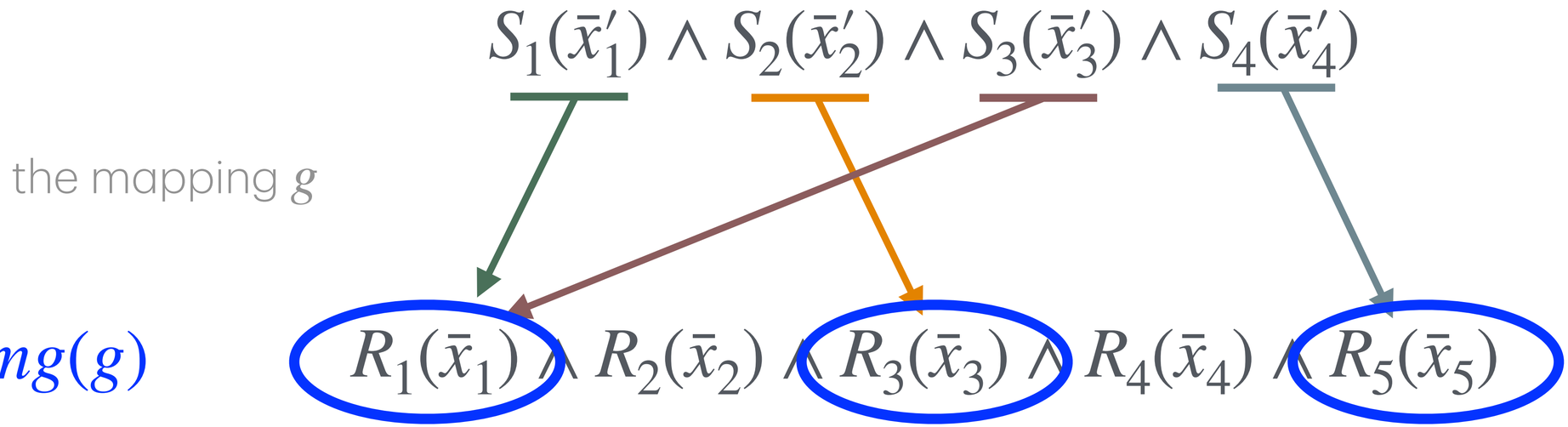
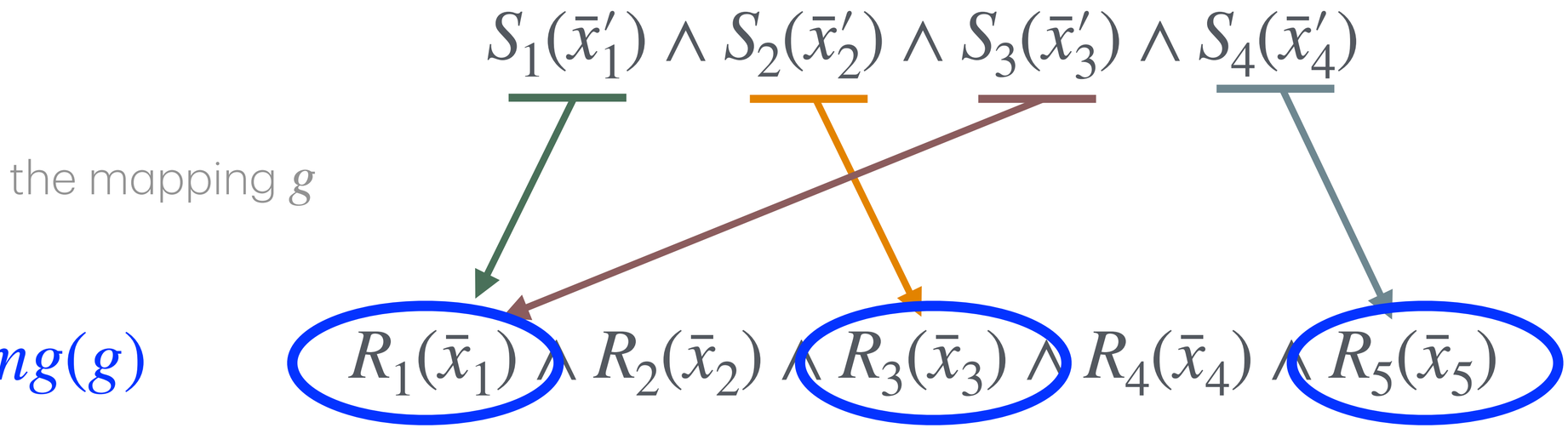
Let $Img(g) = \{R_{i_1}(\bar{x}_{i_1}), R_{i_2}(\bar{x}_{i_2}), \ldots, R_{i_\ell}(\bar{x}_{i_\ell})\}$ be the set of all such images of the mapping g applied to the terms of q' and observe that $|Img(g)| \leq j < k$.



Minimisation by Deletion

Let us define the query $q'' = \{ \bar{y} \mid \exists \bar{z}\ R_{i_1}(\bar{x}_{i_1}) \wedge R_{i_2}(\bar{x}_{i_2}) \wedge \cdots \wedge R_{i_\ell}(\bar{x}_{i_\ell}) \}$ consisting of the terms in $Img(g)$. We see that q'' can be obtained by simply deleting some terms from q.



$$q'' = \{\bar{y} \mid R_1(\bar{x}_1) \wedge R_3(\bar{x}_3) \wedge R_5(\bar{x}_5)$$

Minimisation by Deletion

We use the Homomorphism Theorem to show that q'' is also equivalent to q:

- There is a trivial homomorphism $\mathsf{Tbl}(q'') \to \mathsf{Tbl}(q)$:
  simply map every variable to itself.

$$q'' : \quad R_1(\bar{x}_1) \wedge R_3(\bar{x}_3) \wedge R_5(\bar{x}_5)$$

$$q : \quad R_1(\bar{x}_1) \wedge R_2(\bar{x}_2) \wedge R_3(\bar{x}_3) \wedge R_4(\bar{x}_4) \wedge R_5(\bar{x}_5)$$

Minimisation by Deletion

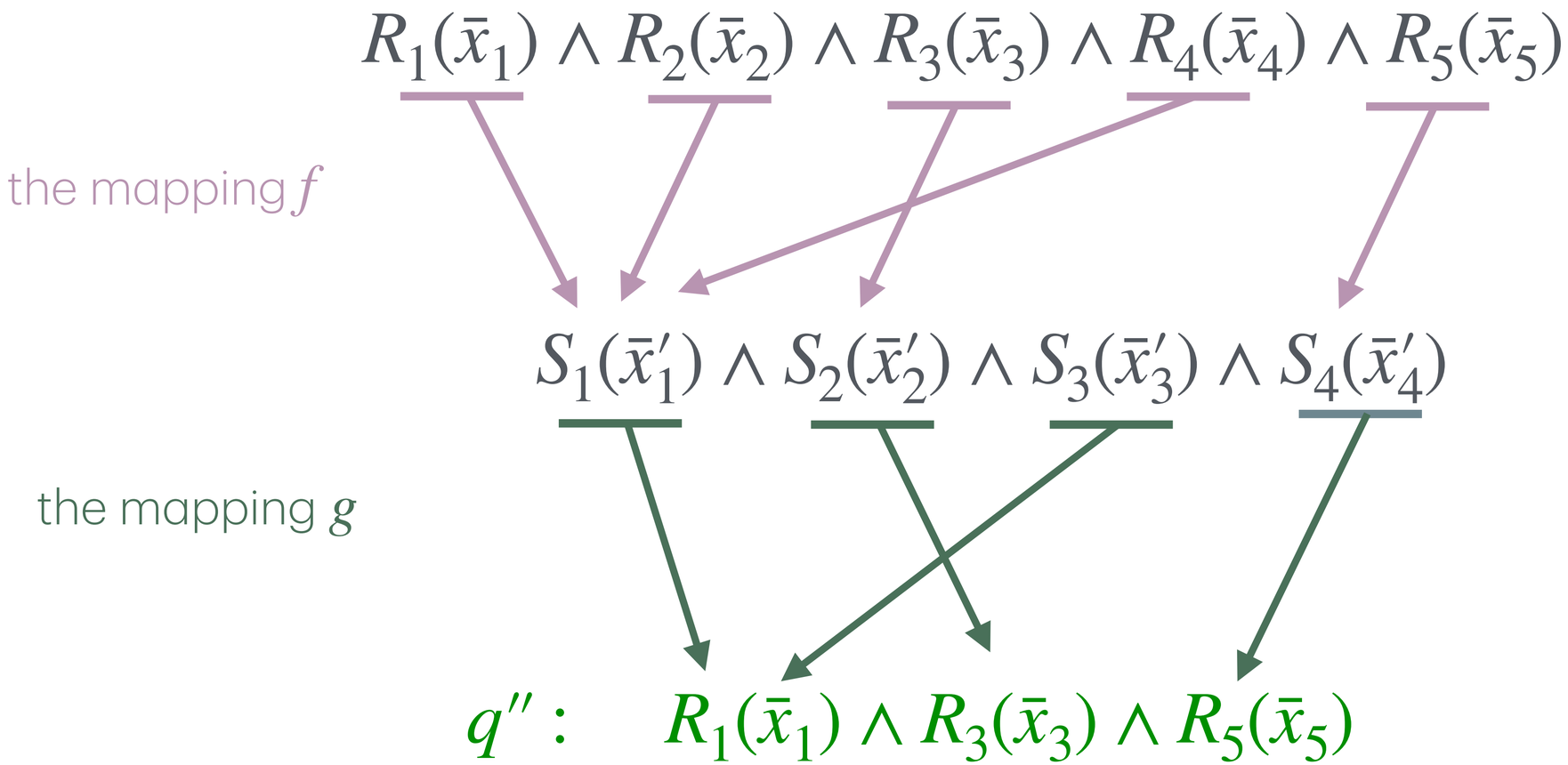
We use the Homomorphism Theorem to show that q'' is also equivalent to q:

- There is a trivial homomorphism $\mathsf{Tbl}(q'') \to \mathsf{Tbl}(q)$:
  simply map every variable to itself.

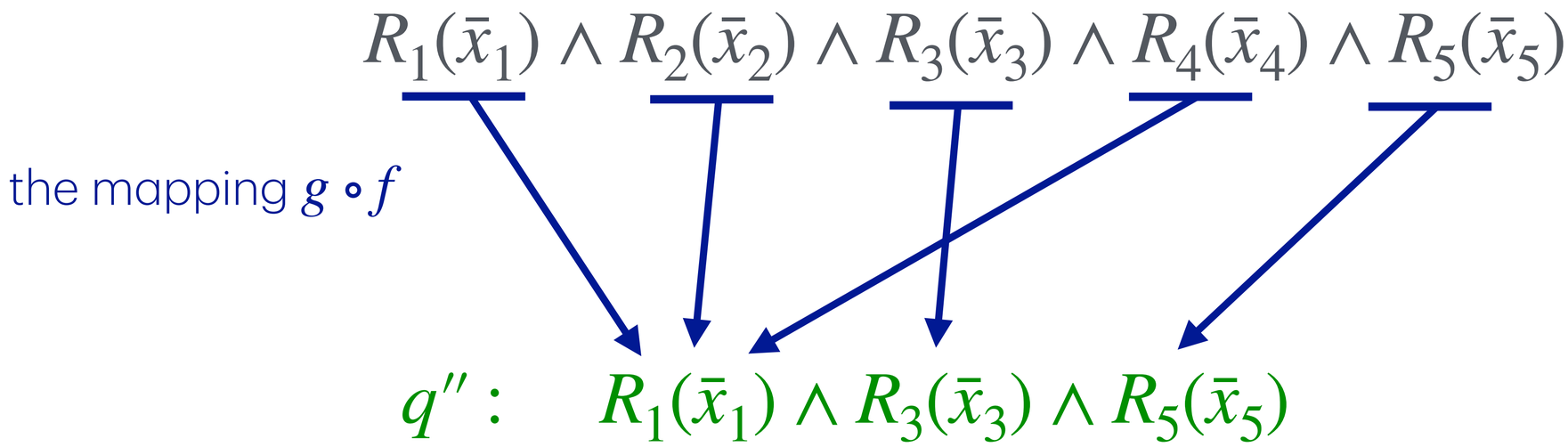
- $g \circ f$ — i.e., the function composition $g(f(\,\cdot\,))$ — is a homomorphism $\mathsf{Tbl}(q) \to \mathsf{Tbl}(q'')$:
  - function f maps every $R_i(\bar{x}_i)$ in q into an $S_\alpha(\bar{x}_\alpha)$ in q' by definition,
  - function g maps every $S_\alpha(\bar{x}_\alpha)$ in q' into an atom of q'' by construction.

✦ $g \circ f$ — i.e., the function composition $g(f(\,\cdot\,))$ — is a homomorphism $\mathsf{Tbl}(q) \to \mathsf{Tbl}(q'')$:

- function f maps every $R_i(\bar{x}_i)$ in q into an $S_\alpha(\bar{x}_\alpha)$ in q' by definition,
- function g maps every $S_\alpha(\bar{x}_\alpha)$ in q' into an atom of q'' by construction.



- $g \circ f$ — i.e., the function composition $g(f(\,\cdot\,))$ — is a homomorphism $\mathsf{Tbl}(q) \to \mathsf{Tbl}(q'')$:
  - function f maps every $R_i(\bar{x}_i)$ in q into an $S_\alpha(\bar{x}_\alpha)$ in q' by definition,
  - function g maps every $S_\alpha(\bar{x}_\alpha)$ in q' into an atom of q'' by construction.



Minimisation by Deletion

Lemma

Assume CQ $q = \{ \bar{y} \mid \exists \bar{z}\ R_1(\bar{x}_1) \wedge R_2(\bar{x}_2) \wedge \cdots \wedge R_k(\bar{x}_k) \}$.

Furthermore, assume q has a semantically equivalent CQ $q' = \{ \bar{y'} \mid \exists \bar{z'}\ S_1(\bar{x'}_1) \wedge S_1(\bar{x'}_1) \wedge \cdots \wedge S_j(\bar{x'}_j) \}$.

Then q is also semantically equivalent to a CQ q'' that can obtained by deleting atoms from q.

Interesting consequence: there is always a unique minimal equivalent query.

We call this minimal equivalent subquery of q the core of q.

An Algorithm for Minimisation

Algorithm 1: Query Minimisation

Input: Conjunctive query q
Result: A minimal conjunctive query q' with $q \equiv q$
$q' \leftarrow q$
repeat
  for *Term* $R(\bar{x})$ *in* q' **do**
    $q''q'$ without $R(\bar{x})$
    if *there is a homomorphism* $\mathsf{Tbl}(q') \to \mathsf{Tbl}(q'')$ **then**
      $q' \leftarrow q''$
    end
  end
until *no change to* q'
return q'

In plain text

Delete terms from the CQ as long as there is still a homomorphism to the query after deletion.

Once this is no longer possible, the minimum is reached.

CQ Minimisation Example

$$q = \{(x, y, z) \mid \exists \alpha\beta\gamma\ R(x, y, \alpha) \wedge R(\beta, y, \gamma) \wedge R(\beta, y, z)\ \}$$

Tbl(q)

A	*B*	*C*
x	y	α
β	y	ɣ
β	y	z

Homomorphism?

$\longrightarrow$

Tbl(q)$\setminus\{(x, y, \alpha)\}$

A	*B*	*C*
x	y	α
β	y	ɣ
β	y	z

No, because $h(x) = x$, the first row can't be mapped into the right-hand tableau.

CQ Minimisation Example

$$q = \{(x, y, z) \mid \exists \alpha\beta\gamma\ R(x, y, \alpha) \wedge R(\beta, y, \gamma) \wedge R(\beta, y, z)\ \}$$

Tbl(q)

A	*B*	*C*
x	y	α
β	y	ɣ
β	y	z

Homomorphism?

$\longrightarrow$

Tbl(q)$\setminus\{(\beta, y, \gamma)\}$

A	*B*	*C*
x	y	α
β	y	ɣ
β	y	z

Yes! x, y, z, β map to themselves and $h(\gamma) = z$

CQ Minimisation Example

$$q' = \{(x, y, z) \mid \exists \alpha\beta \; R(x, y, \alpha) \wedge R(\beta, y, z) \; \}$$

Tbl(q')

A	*B*	*C*
x	y	α
β	y	z

Homomorphism?

$\longrightarrow$

A	*B*	*C*
x	y	α
β	y	z

or

A	*B*	*C*
x	y	α
β	y	z

Both times no.

Hence, q' is minimal!

Complexity of CQs

Data vs Combined Complexity

Recall, in classical computational complexity theory we study algorithm behavior (time/space/etc.) relative to the size of the input.

In query answering problems there are different variants of this problem:

Eval(q, D)

Input size is the sum of the query size and database size

Matches natural settings such as a DBMS, where queries and data come from user and are arbitrary.

q-Eval(D)

Input size is only the database!

Motivated by the fact that queries are usually much smaller than the databases.

Data vs Combined Complexity

Recall, in classical computational complexity theory we study algorithm behavior (time/space/etc.) relative to the size of the input.

In query answering problems there are different variants of this problem:

Combined Complexity

Input size is the sum of the query size and database size

Matches natural settings such as a DBMS, where queries and data come from user and are arbitrary.

Data Complexity

Input size is only the database!

Motivated by the fact that queries are usually much smaller than the databases.

Our Focus Now

CQ-EVAL

Input: Conjunctive query q, database D (of same schema)

Output: $q(D) \neq \emptyset$

Recall, this corresponds to combined complexity.

NP-Membership

$$\{\, \bar{y} \mid \exists \bar{z}\, R_1(\bar{x}_1) \wedge R_2(\bar{x}_2) \wedge \cdots \wedge R_k(\bar{x}_k) \,\}$$

When is $q(D) \neq \emptyset$?

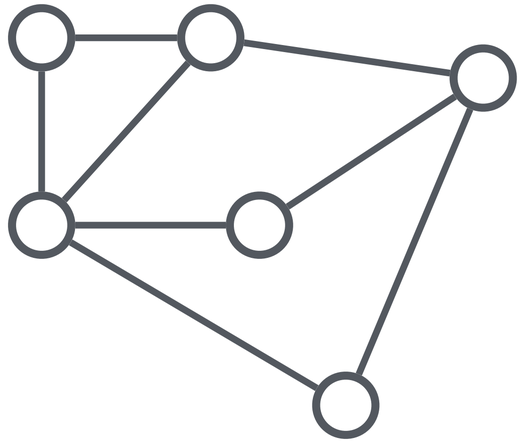
If there is any homomorphism from $\mathsf{Tbl}^*(q)$ to D.

NP-membership is straightforward: guess and check a homomorphism.

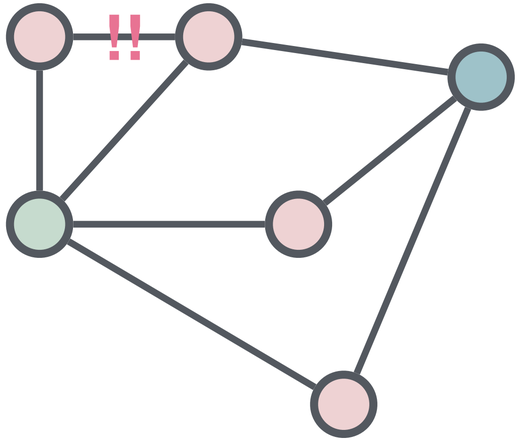
NP-Hardness

- There is an easy reduction from **3-Colourability.**
- **3-Colourability** takes a graph Gas input and decides whether G is 3-colourable.

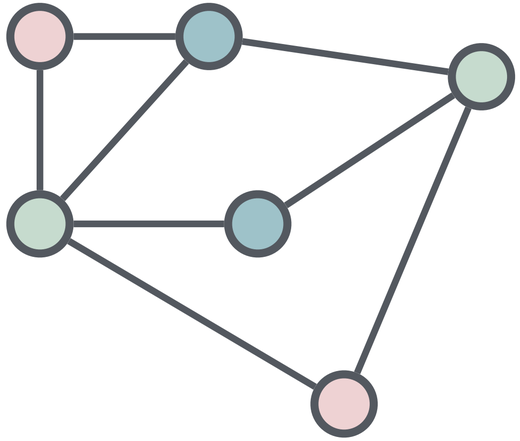
  That is, can we color the vertices of G with red, green, and blue such that no edge is between two vertices of the same colour?



Not a 3-colouring

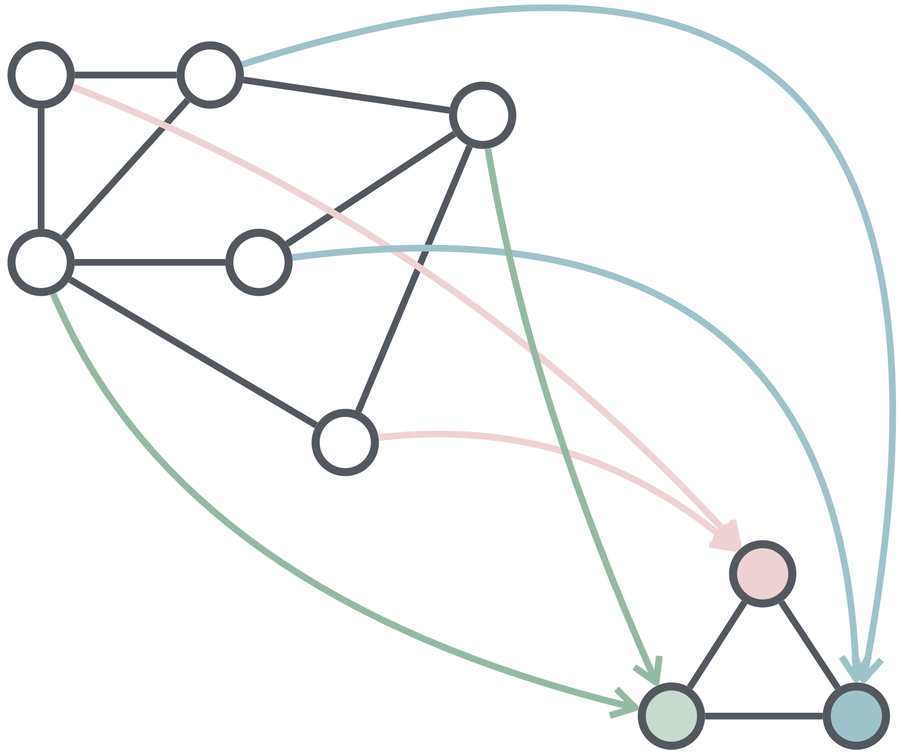


Valid 3-colouring



NP-Hardness

- **3-Colourability** is equivalent to having a homomorphism into the triangle graph.
- The three nodes of the triangle intuitively represent the three colours.
- Note that if there is an edge between v and u, then v, u can't be mapped to the same vertex, i.e., adjacent vertices can't be mapped to the same colour.

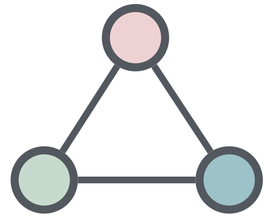


NP-Hardness

This homomorphism into the triangle can be trivially expressed as a conjunctive query.

- Take an input for **3-Colourability**, i.e., a graph G.
- Create a database with relation E for the triangle:

A	*B*
red	green
green	red
red	blue
blue	red
green	blue
blue	green



- Encode the graph as a conjunctive query:

$$q = \{\ () \mid \exists \bar{v} \bigwedge_{\{v_i, v_j\} \in E(G)} E(v_i, v_j) \wedge E(v_j, v_i)\ \}$$

- There is a homomorphism $\mathsf{Tbl}^*(q) \to D$ if and only if G is 3-colourable.

Complexity of CQ -Eval

Theorem

CQ-Eval is **NP**-complete in combined complexity. Moreover, the **NP**-hardness holds already for schemas with a single binary relation symbol.

Complexity of CQ Containment

Recall the Homomorphism Theorem:

$$q_1 \subseteq q_2 \iff \mathsf{Tbl}(q_2) \xrightarrow{hom} \mathsf{Tbl}(q_1)$$

Same reduction applies here too: check whether the query q_1 that represents the triangle is contained in the query q_2 that represents graph G.

Theorem
Deciding CQ Containment is **NP**-complete.

Complexity of CQ Minimisation

Theorem

Checking whether a query q is minimal is **co-NP**-complete.

Intuition: We need to check whether there are *no homomorphisms* into any query obtained by deleting atoms.