Full Proofs for Submission #1565

Additional definitions and lemmas

Definition 15. *Let R be a relation on a set X. Then we write R^{-1} for the* inverse relation *defined by $xR^{-1}y$ iff yRx.*

Observe that R^{-1} is a total order if and only if R is a total order and R^{-1} is a linear order if and only if R is a linear order.

Lemma 16. *Let X be a set of objects and $\mathcal{X} \subseteq \mathcal{P}(X)\setminus\{\emptyset\}$ a family of sets. Assume that $\mathcal{X}$ is DI^S-orderable with respect to a linear order $\leq$. Then $\mathcal{X}$ is DI^S-orderable with respect to $\leq^{-1}$.*

Similarly, if we assume that $\mathcal{X}$ is DI-orderable with respect to a linear order $\leq$, then $\mathcal{X}$ is DI-orderable with respect to $\leq^{-1}$.

Proof. Let $\preceq$ be an order on $\mathcal{X}$ that satisfies dominance and strict independence with respect to $\leq$. Then we claim that $\preceq^{-1}$ satisfies dominance and strict independence with respect to $\leq^{-1}$. Assume $A, A \cup \{x\} \in \mathcal{X}$, then $\forall y \in A(y <^{-1} x)$ implies $\forall y \in A(y > x)$, which implies $A \cup \{x\} \prec A$ by assumption, hence $A \prec^{-1} A \cup \{x\}$. Similarly, $\forall y \in A(x <^{-1} y)$ implies $A \cup \{x\} \prec^{-1} A$.

Now, assume $A, B, A\cup\{x\}, B\cup\{x\} \in \mathcal{X}$ and $A \prec^{-1} B$. Then $B \prec A$ and hence by assumption $B \cup \{x\} \prec A \cup \{x\}$ which implies $A \cup \{x\} \prec^{-1} B \cup \{x\}$.

The argumentation for independence is the same. □

The following definition is taken from (Murray and Williams 2017).

Definition 17. *An algorithm $R : \{0,1\}^* \times \{0,1\}^* \to \{0,1,\star\}$ is a* polylog-time reduction *from L to L' if there are constants $c \geq 1$ and $k \geq 1$ such that for all $x \in \{0,1\}^*$,*

- *$R(x,i)$ has random access to x and runs in $O((\log(|x|))^k)$ time for all $i \in \{0,1\}^{\lceil 2c\log(|x|)\rceil}$*
- *there is an $l_x \leq |x|^c + c$ such that $R(x,i) \in \{0,1\}$ and for all $i \leq l_x$ and $R(x,i) = \star$ for all $i > l_x$.*
- *$x \in L$ iff $R(x,1) \cdot R(x,2) \cdots R(x,l_x) \in L'$.*

Here $\cdot$ is the string concatenation and $\star$ is the out of bounds character that marks the end of a string.

Proof of Proposition 4

Let ϕ be a instance of SAT with n variables and m clauses. Then we produce an instance $(X, \mathcal{X})$ of STRONG DI^S-ORDERABILITY. We produce this instance in a way that there is a linear order $\leq$ such that $\mathcal{X}$ is DI^S orderable with respect to $\leq$ only if ϕ is satisfiable. Then, we sketch how to use a satisfying assignment of ϕ to construct for any linear order $\leq'$ on X a linear order $\preceq$ on $\mathcal{X}$ that satisfies dominance and strict independence.

The set of elements X contains for every variable V_i elements $x^-_{i,1}, x^-_{i,2}, x^+_{i,1}$ and $x^+_{i,2}$. Furthermore, it contains for every clause C_i variables $z^a_i, y^a_i, \min^a_i$ and $\max^a_i$ for $a \in \{1,2,3\}$. We call the elements $\min^a_i$ and $\max^a_i$ the extremum-elements. Finally, it contains two elements v_1 and v_2. Then we define the following linear order $\leq$ on X:

$$\begin{aligned}\min^1_1 < \min^2_1 < \cdots < \min^3_m < x^-_{1,1} < x^-_{1,2} < \dots \\ < x^-_{n,2} < v_1 < v_2 < z^1_1 < z^2_1 \cdots < z^3_m < \\ y^1_1 < y^2_1 < \cdots < y^3_m < x^+_{1,1} < x^+_{1,2} < \dots \\ < x^+_{n,2} < \max^1_1 < \max^2_1 < \cdots < \max^3_m\end{aligned}$$

Next, we will construct a family $\mathcal{X}$ that is only DI^S-orderable with respect to $\leq$ if ϕ is satisfiable. In the following, we write

$$Y := \{x \in X \mid v_1 \leq x \leq y^m_3\}$$

First, we add for every variable V_i sets $X^1_i = Y \cup \{x^-_{i,1}, x^+_{i,1}\}$ and $X^2_i = Y \cup \{x^-_{i,2}, x^+_{i,2}\}$ We call these the class 1 sets and write Cl_1 for the collection of all class 1 sets.

Intuitively, the truth value of V_i will be coded by the preference between X^1_i and X^2_i, where $X^1_i \prec X^2_i$ equals V_i is false and $X^2_i \prec X^1_i$ equals V_i is true.

We will now add for every clause new sets that lead to a contradiction if the clause is not satisfied. Let C_i be a clause with variables V_j, V_k, V_l. We add

$$\begin{aligned}X^1_j \setminus \{y^1_i\}, X^2_j \setminus \{y^1_i\}, X^1_k \setminus \{y^2_i\}, \\ X^2_k \setminus \{y^2_i\}, X^1_l \setminus \{y^3_i\} \text{ and } X^2_l \setminus \{y^3_i\}.\end{aligned}$$

We call these the class 2 sets and write Cl_2 for the collection of all class 2 sets. By "reverse strict independence"[4]we know that the preference between $X^1_j\setminus\{y^1_i\}$ and $X^2_j\setminus\{y^1_i\}$ must be the same as the preference between X^1_j and X^2_j. The same holds for the other two variables. Now, if all variables occur positively in C_i, we add sets such that $X^2_j \setminus \{y^1_i\} \prec X^1_k \setminus \{y^2_i\}, X^2_k \setminus \{y^2_i\} \prec X^1_l \setminus \{y^3_i\}$ and $X^2_l \setminus \{y^3_i\} \prec X^1_j \setminus \{y^1_i\}$ must hold in any order $\preceq$ on $\mathcal{X}$ that satisfies dominance and strict independence with respect to $\leq$. We call this enforcing these preferences. Then we get a contradiction if V_j, V_k and V_l are false because

$$\begin{aligned}X^1_j \setminus \{y^1_i\} \prec X^2_j \setminus \{y^1_i\} \prec X^1_k \setminus \{y^2_i\} \prec \\ X^2_k \setminus \{y^2_i\} \prec X^1_l \setminus \{y^3_i\} \prec X^2_l \setminus \{y^3_i\} \prec X^1_j \setminus \{y^1_i\}\end{aligned}$$

holds. If a variable, say V_j, occurs negatively in C_i, we switch X^1_j and X^2_j and enforce $X^1_j \setminus \{y^1_i\} \prec X^1_k \setminus \{y^2_i\}$ and $X^2_l \setminus \{y^3_i\} \prec X^2_j \setminus \{y^1_i\}$.

Next, we show how we can enforce these preference. Assume we want to enforce $X^a_j \setminus \{y^1_i\} \prec X^b_k \setminus \{y^2_i\}$ for $a, b \in \{1,2\}$. We add $\{z^1_i\}, \{z_i, \max^1_i\}$ and $(X^a_j \setminus \{y^1_i\}) \cup \{\max^1_i\}$. Our goal is to enforce $(X^a_j \setminus \{y^1_i\}) \cup \{\max^1_i\} \prec \{z^1_i, \max^1_i\}$ which forces by reverse strict independence $X^a_j \setminus \{y^1_i\} \prec \{z^1_i\}$. Then we enforce $\{z^1_i\} \prec X^b_k \setminus \{y^2_i\}$ to get by transitivity $X^a_j \setminus \{y^1_i\} \prec X^b_k \setminus \{y^2_i\}$ as desired. To enforce $(X^a_j \setminus$

[4]Every linear order satisfying strict independence has to satisfy reverse strict independence, i.e. $A \cup \{x\} \prec B \cup \{x\}$ implies $A \prec B$: Assume otherwise $B \prec A$ holds, then by strict independence $B \cup \{x\} \prec A \cup \{x\}$ must hold, contradicting $A \cup \{x\} \prec B \cup \{x\}$. Hence by the totality of $\preceq$ we have $A \prec B$.

$\{y_i^1\}) \cup \{\max_i^1\} \prec \{z_i, \max_i^1\}$ we add a sequence of sets $A_1, A_2, \dots, A_k$ such that $A_1 = (X_j^a \setminus \{y_i^1, z_i^1\}) \cup \{\max_i^1\}$, $A_{i+1} = A_i \setminus \min_{\leq}(A_i)$ and $A_k = \{\max_i^1\}$. This enforces by dominance $A_1 \prec A_2 \prec \cdots \prec A_k$ which enforces by transitivity

$$A_1 = (X_j^a \setminus \{y_i^1, z_i^1\}) \cup \{\max_i^1\} \prec \{\max_i^1\} = A_k.$$

Finally, this enforces by strict independence the desired $(X_j^a \setminus \{y_i^1\}) \cup \{\max_i^1\} \prec \{z_i, \max_i^1\}$. Using the same idea and $\min_i^1$ we enforce $\{z_i^1\} \prec X_k^b \setminus \{y_i^2\}$ finishing the construction for $X_j^a \setminus \{y_i^1\} \prec X_k^b \setminus \{y_i^2\}$. We enforce $X_k^2 \setminus \{y_i^2\} \prec X_l^1 \setminus \{y_i^3\}$ and $X_l^2 \setminus \{y_i^3\} \prec X_j^1 \setminus \{y_i^1\}$ similarly using $z_i^2, \max_i^2$ and $\min_i^2$ resp. $z_i^3, \max_i^3$ and $\min_i^3$.

We repeat this procedure for every clause. We call the sets added in this step the class 3 sets and write Cl_3 for the collection of all class 3 sets. Now, by construction, $\mathcal{X}$ can only be DI^S-orderable with respect to $\leq$ if ϕ is a positive instance of SAT.

Next, we pick an arbitrary linear order $\leq'$ on X. We distinguish two cases $v_1 <' v_2$ and $v_2 <' v_1$. By Lemma 16 it suffices to show DI^S orderability in the first case, because $v_2 <' v_1$ implies $v_1 <'^{-1} v_2$ and we only need to show DI^S orderability for one of these two orders. Hence, we can assume in the following w.l.o.g. $v_1 <' v_2$. Now, we want to construct an order $\preceq$ on $\mathcal{X}$ that satisfies dominance and strict independence with respect to $\leq'$ if ϕ is satisfiable.

First, we order the sets X_i^1 and X_i^2 according to a satisfying assignment of ϕ, i.e. $X_i^1 \prec X_i^2$ if V_i is false in the assignment and $X_i^2 \prec X_i^1$ if it is true. Then, we project this order down to the class 2 sets by reverse strict independence. Finally, we take the transitive closure of this order. It is clear by construction, that this is an acyclic strict partial order if and only if ϕ is satisfiable. Now, for any clause C_i, we fix any linear order on the sets

$$X_j^1 \setminus \{y_i^1\}, X_j^2 \setminus \{y_i^1\}, X_k^1 \setminus \{y_i^2\}, \\ X_k^2 \setminus \{y_i^2\}, X_l^1 \setminus \{y_i^3\} \text{ and } X_l^2 \setminus \{y_i^3\}.$$

that extends this order.

For the class 1 sets we have ordered all pairs (X_i^1, X_i^2) but we still have to fix an order between these pairs. For the class 2 sets, we have fixed an order on between all sets introduced for a single clause, but we have to fix an order between sets from different clauses. Now, we observe that $A, A \cup \{x\} \in Cl_1 \cup Cl_2$ implies that $A \in Cl_1$, $B \in Cl_2$ and $x = y_i^a$ for $j \leq m$ and $a \leq 3$ as all Class 1 sets differ from all other class 1 sets in at least two elements and all class 2 sets differ from all other class 2 sets in at least two elements. Hence the only possible application of strict independence on class 1 and 2 is the one already covered by construction. Dominance is applicable only if y_i^a for some i and a is the minimal or maximal element of the set it gets removed from. We fix an order on the pairs and clauses that is compatible with these applications of dominance. First, assume the minimal element $y_{i^-}^{a^-}$ of the form y_i^a and the maximal element $y_{i^+}^{a^+}$ of the form y_i^a are used for the same clause. Let X_j^b and X_k^c be the sets such that $X_j^b \setminus \{y_{i^-}^{a^-}\} \in \mathcal{X}$ and $X_k^c \setminus \{y_{i^+}^{a^+}\} \in \mathcal{X}$ holds. Then, by construction $j \neq k$. In that case we fix any linear order $\leq''$ on the pairs (X_i^1, X_i^2) such that $(X_j^1, X_j^2) \leq'' (X_k^1, X_k^2)$ holds and an arbitrary order on the clauses. We set X_i^a is smaller than any class 2 set if $(X_i^1, X_i^2) <'' (X_j^1, X_j^2)$. Furthermore, any set X_i^a is bigger than any class 2 set if $(X_j^1, X_j^2) <'' (X_i^1, X_i^2)$. This is obviously a linear order and we have $X_j^b \prec X_j^b \setminus \{y_{i^-}^{a^-}\}$ and $X_k^c \setminus \{y_{i^+}^{a^+}\} \prec X_k^c$ for $b, c \leq 2$. Hence the constructed order on $Cl_1 \cup Cl_2$ satisfies dominance.

Now, assume the minimal element $y_{i^-}^{a^-}$ of the form y_i^a and the maximal element $y_{i^+}^{a^+}$ of the form y_i^a are used for different clauses C_{i^-} and C_{i^+}. We fix any order on the clauses such that Cl_{i^+} is smaller than Cl_{i^-} and an arbitrary order on the pairs. Additionally we set all sets from clauses smaller or equal Cl_{i^+} smaller than any set from a class 1 and any set from a clause larger that Cl_{i^+} larger than any set from class 1. This is obviously a linear order and we have $X_j^b \prec X_j^b \setminus \{y_{i^-}^{a^-}\}$ and $X_k^c \setminus y_{i^+}^{a^+} \prec X_k^c$ for $b, c \leq 2$. Hence the constructed order on $Cl_1 \cup Cl_2$ satisfies dominance.

Next, we add the class 3 sets. First, we observe that if we have a set $A \in Cl_1 \cup Cl_2$ and $A \cup \{x\}$ then we know $A \in Cl_2$ and x is an extremum-element. On the other hand, there is no set $A \in Cl_3$ such that $A \cup \{x\} \in Cl_1 \cup Cl_2$ holds, as every set in C_3 either contains an extremum-element or it is a singleton and no set in class 1 and 2 contains an extremum-element and every set in class 1 and 2 has more than three elements.

Hence, for the interaction of class 3 with the other classes, we only have to consider dominance if we add an extremum element to a class 2 set that is smaller/larger than any element already in the class 2 set. This is achieved by the following construction. Let A be a set in Cl_3 containing an extremum-element mm. Then A is in Cl_3^+ if $v_1 <' mm$ holds and in Cl_3^- otherwise. Then we set $A \prec B$ if

- $A \in Cl_1 \cup Cl_2$ and $B \in Cl_3^+$,
- $A \in Cl_3^-$ and $B \in Cl_1 \cup Cl_2$.

Next, we order the sets in Cl_3^+ and Cl_3^-. First, we define an order between sets containing the same extremum element. In the following, we write for a set A that contains an extremum-element mm (which is, by construction unique) $A_S := \{x \in A \mid x <' mm\}$ for the set of elements in A that are smaller than mm and $A_L := \{x \in A \mid mm <' x\}$ for the set of elements in A that are larger than mm

We set $A \prec B$ for sets A, B that both contain the same extremum element of the form $\max_i^c$ if:

- $\max_{\leq'}(A_L \triangle B_L) \in B$,
- $A_L = B_L$ and $\min_{\leq'}(A_S \triangle B_S) \in A$.

Here, $\triangle$ is the symmetric difference operator, i.e. $A \triangle B := (A \cup B) \setminus (A \cap B)$. We claim that this order satisfies dominance and strict independence. It satisfies strict independence because for all sets S, T by definition $S \cup \{x\} \triangle T \cup \{x\} = S \triangle T$ for any $x \notin S \cup T$. For dominance, assume $x <' \min_{<'}(A)$ and $\max_i^c \in A, A \cup \{x\}$. Then $A_L = (A \cup \{x\})_L$ and $\min_{\leq'}(A_S \triangle (A \cup \{x\})_S) = x$ Hence, $A \cup \{x\} \prec A$. The case $\max_{<'}(A) <' x$ is similar.

We observe that we may have either $X^a_j \setminus \{y^b_i\} \cup \{\max^c_i\} \prec \{z^c_i, \max^c_i\}$ or $\{z^c_i, \max^c_i\} \prec X^a_j \setminus \{y^b_i\} \cup \{\max^c_i\}$. In the first case, we add z^c_i in the order exactly after $X^a_j \setminus \{y^b_i\}$ and in the second case exactly before $X^a_j \setminus \{y^c_i\}$.

Now let $X^d_k \setminus \{y^e_i\}$ be the set for which we enforce the preference $X^a_j \setminus \{y^c_i\} \prec X^d_k \setminus \{y^e_i\}$. Then, this construction implies $\{z^c_i\} \prec X^d_k \setminus \{y^e_i\}$. Therefore, we have to make sure that $\{z^c_i, \min^c_i\} \prec X^d_k \setminus \{y^e_i\} \cup \{\min^c_i\}$ holds as intended by the construction to avoid a contradiction. For this we use the fact that $v_1 <' v_2$ holds. We set $A \prec B$ for elements A, B if they both contain an element of the form $\min^c_i$ if:

- $v_2 \in B$ and $v_2 \notin A$ $(\star)$,
- $v_2 \in A, B$ or $v_2 \notin A, B$ and $\max_{\leq'}(A_L \triangle B_L) \in B$,
- $v_2 \in A, B$ or $v_2 \notin A, B$, $A_L = B_L$ and $\min_{\leq'}(A_S \triangle B_S) \in A$.

It is clear that $(\star)$ implies $\{z^c_i, \min^c_i\} \prec X^d_k \setminus \{y^e_i\} \cup \{\min^c_i\}$. It is also clear that it satisfies strict independence because the $(\star)$ implies a preference between sets $A \cup \{x\}$ and $B \cup \{x\}$ for $x \notin A \cup B$ iff it implies the same preference for A and B. If $(\star)$ is not applicable, strict independence is satisfied by the same argument as above. Now, for dominance $v_2 \in (A \triangle (A \cup \{x\}))$ implies $x = v_2$. Then, $x < \min_{<'}(A)$ is not possible because by construction $v_1 \in A$ holds and we assume $v_1 <' v_2$. If we have $\max_{<'}(A) <' x$ then dominance is satisfied because $A \prec A \cup \{x\}$ holds by $(\star)$. If $x \neq v_2$, then $(\star)$ is not applicable and dominance is satisfied by the same argument as above.

We observe that this is the only application of strict independence with sets from class 3 and sets not from class 3 because if we have a set $A \in Cl_1 \cup Cl_2$ and $A \cup \{x\} \in Cl_3$ there is no other set $B \in Cl_1 \cup Cl_2$ such that $B \cup \{x\} \in \mathcal{X}$ holds. Finally, we have to extend the order $\preceq$ to the whole class 3. However, any two sets not yet comparable differ in at least two elements. Hence, any completion of $\preceq$ satisfies dominance and strict independence. $\square$

Proof of Theorem 5

Π^p_2-membership is clear as we can universally guess a linear order $\leq$ on X and then check via the NP-oracle if $\mathcal{X}$ is DI^S-orderable with respect to $\leq$.

It remains to show that STRONG DI^S-ORDERABILITY is Π^p_2-hard. We do this by extending the reduction above to a reduction from a Π_2-SAT instance $\phi = \forall \vec{W} \exists \vec{V} \psi(\vec{W}, \vec{V})$. Let $w_1 \dots w_l$ be the universally quantified variables. Then X contains the same elements as in the construction above and additionally for every universally quantified variable w_i elements w^1_i and w^2_i as well as elements $y^q_i, \min^q_i, \max^q_i$ and $\overline{y^q_i}, \overline{\min^q_i}, \overline{\max^q_i}$. Then we add the same sets as in the reduction above, except that the elements w^1_i, w^2_i, y^q_i and $\overline{y^q_i}$ are included in the set Y used in the construction. Furthermore, we add for every universally quantified variable w_i sets $X^1_i \setminus \{y^q_i\}$, $X^2_i \setminus \{y^q_i\}$ and $\{w^1_i, w^2_i\}$ Furthermore, we enforce as described above $X^1_i \setminus \{y^q_i\} \prec \{w^1_i\}$ and $\{w^2_i\} \prec X^2_i \setminus \{y^q_i\}$ using $\min^q_i$ and $\max^q_i$. Now, let $\leq'$ be a linear order on X such that $w^1_i <' w^2_i$ holds. Then $X^1_i \prec X^2_i$ must hold for every order $\preceq$ on $\mathcal{X}$ that satisfies dominance and strict independence with respect to $\leq'$. Now, we add additionally sets $X^1_i \setminus \{\overline{y^q_i}\}$ and $X^2_i \setminus \{\overline{y^q_i}\}$. Then, we enforce $X^1_i \setminus \{\overline{y^q_i}\} \prec \{w^2_i\}$ and $\{w^1_i\} \prec X^2_i \setminus \{\overline{y^q_i}\}$ using $\overline{\min^q_i}$ and $\overline{\max^q_i}$. Analogously to above $X^2_i \prec X^1_i$ must hold for every order $\preceq$ on $\mathcal{X}$ that satisfies dominance and strict independence with respect to a linear order $\leq''$ on X such that $w^2_i <'' w^1_i$ holds.

We claim that we $(X, \mathcal{X})$ can only be a positive instance of STRONG DI^S-ORDERABILITY, if ϕ is a positive instance of Π_2-SAT. First, we fix the same order $\leq$ as above on the elements that occur already in the first reduction. Then, for every truth assignment T to the variables in $\vec{W}$ there is a linear order $\leq^*$ on X that coincides with $\leq$ on the old elements such that $w^1_i <^* w^2_i$ if w_i is assigned false in T and $w^2_i <^* w^1_i$ if w_i is assigned true in T. Now, if there is no satisfying assignment on ϕ that extends T, then there can be no order on $\mathcal{X}$ satisfying dominance and strict independence with respect to $\leq^*$. Hence $(X, \mathcal{X})$ can only be DI^S orderable with respect to every linear order $\leq^*$ if ϕ is a positive instance of Π_2-SAT.

It remains to show that if ϕ is satisfiable then $(X, \mathcal{X})$ is a positive instance of STRONG DI^S-ORDERABILITY. This can be done using nearly the same construction as above treating $X^1_i \setminus \{y^q_i\}$ and $X^2_i \setminus \{y^q_i\}$ as Class 2 sets, all other new sets as Class 3 sets and inserting $\{w^1_i\} \prec \{w^1_i, w^2_i\} \prec \{w^2_i\}$ resp. $\{w^2_i\} \prec \{w^1_i, w^2_i\} \prec \{w^1_i\}$ where we would insert z^j_i. The only exception has to be made if there is an i such that $y^q_i = \min(X^1_i)$ and $\overline{y^q_i} = \max(X^1_i)$ or $\overline{y^q_i} = \min(X^2_i)$ and $y^q_i = \max(X^2_i)$. In the first case, we set $A \prec B$ for the sets containing $\overline{\min^q_i}$ if:

- $y^q_i \in A$ and $y^q_i \notin B$
- $y^q_i \in A, B$ or $y^q_i \notin A, B$ and $\max(A_L \triangle B_L) \in B$,
- $y^q_i \in A, B$ or $y^q_i \notin A, B$, $A_L = B_L$ and $\min(A_S \triangle B_S) \in A$.

where $A_L := \{x \in A \mid \overline{\min^q_i} <' x\}$ and $A_S := \{x \in A \mid x <' \overline{\min^q_i}\}$. And for $A \prec B$ for the sets containing $\max^q_i$ if

- $\overline{y^q_i} \in B$ and $\overline{y^q_i} \notin A$
- $y^q_i \in A, B$ or $y^q_i \notin A, B$ and $\max(A_L \triangle B_L) \in B$,
- $y^q_i \in A, B$ or $y^q_i \notin A, B$, $A_L = B_L$ and $\min(A_S \triangle B_S) \in A$.

where $A_L := \{x \in A \mid \max^q_i <' x\}$ and $A_S := \{x \in A \mid x <' \max^q_i\}$.

It is clear that these orders satisfy dominance and strict independence, similarly to the orders on the class 3 sets defined above. Furthermore, we have $\{w^1_i, \max^q_i\} \prec (X^1_i \setminus \{y^q_i\}) \cup \{\max^q_i\}$ and $(X^1_i \setminus \{\overline{y^q_i}\}) \cup \{\overline{\min^q_i}\} \prec \{w_i, \overline{\min^q_i}\}$ which allows us to set $X^1_i \setminus \{\overline{y^q_i}\} \prec \{w_i\} \prec X^1_i \setminus \{y^q_i\}$ which is consistent with the enforced $X^1_i \setminus \{\overline{y^q_i}\} \prec X^1_i \prec X^2_i \setminus \{y^q_i\}$. The second case can be treated analogously. $\square$

Proof of Theorem 6

We need to change the reduction from a Π_2-SAT instance ϕ in two places compared to Theorem 5. First, we need to modify the way we enforce a strict preference $X_i^a \setminus \{y_j^b\} \prec X_k^c \setminus \{y_j^d\}$ using independence instead of strict independence. We replace every element z_j^b by two elements z_j^b and $\overline{z_j^b}$, set $z_j^b < \overline{z_j^b}$ and add the sets $\{z_j^b\}, \{z_j^b, \overline{z_j^b}\}, \{\overline{z_j^b}\}$ to $\mathcal{X}$. Then, to enforce $X_i^a \setminus \{y_j^b, x_i^-\} \preceq \{z_j^b\}$ we add the same sequence $A_1, \ldots, A_l$ as in the proof of Proposition 4 and, additionally, the set $(X_i^a \setminus \{y_j^b, x_i^-\}) \cup \{\max_j^b\}$. We observe $l > 3$ and that the following preference enforced by dominance

$$A_2 = (X_i^a \setminus \{y_j^b, z_j^b, x_i^-\}) \cup \{\max_j^b\} \prec \{\max_j^b\} = A_l$$

which enforces by independence

$$(X_i^a \setminus \{y_j^b, x_i^-\}) \cup \{\max_j^b\} \preceq \{z_j^b, \max_j^b\}$$

and hence by dominance

$$(X_i^a \setminus \{y_j^b\}) \cup \{\max_j^b\} \prec \{z_j^b, \max_j^b\}.$$

This gives us by "reverse independence"[5]the desired $X_i^a \setminus \{y_j^b, x_i^-\} \preceq \{z_j^b\}$. Now, we can enforce $\{\overline{z_j^b}\} \preceq X_k^c \setminus \{y_j^d\}$ similarly. Then, this enforces by dominance

$$X_i^a \setminus \{y_j^b\} \preceq \{z_j^b\} \prec \{z_j^b, \overline{z_j^b}\} \prec \{\overline{z_j^b}\} \preceq X_k^c \setminus \{y_j^d\}.$$

and hence by transitivity $X_i^a \setminus \{y_j^b\} \prec X_k^c \setminus \{y_j^d\}$.

Second, we have to make sure that all preferences between sets X_i^1 and X_i^2 are strict. We borrow an idea from (Maly and Woltran 2017) to achieve this. We add for every variable X_i new elements ordered in $\leq$ as follows

$$a_i^- < b_i^- < c_i^- < d_i^- < r_i < s_i < d_i^+ < c_i^+ < b_i^+ < a_i^+.$$

such that v_1 and v_2 lie between d_i and r_i in the order $\leq$. Then, we add new sets $A_i := \{a_i^-, v_1, v_2, r_i, s_i, a_i^+\}$, $B_i := \{b_i^-, v_1, v_2, r_i, s_i, b_i^+\}$, $C_i := \{c_i^-, v_1, v_2, r_i, s_i, c_i^+\}$ and $D_i := \{d_i^-, v_1, v_2, r_i, s_i, d_i^+\}$. Now, let z_i^a, $\overline{z_i^a}$, $\max_i^a$ and $\min_i^a$ be new elements where we set $z_i^a, \overline{z_i^a} \in Y$. Then, we enforce with the method described above $A_i \prec X_i^2$ using these new elements. Furthermore, we enforce $X_i^1 \prec B_i$, $X_i^2 \prec C_i$ and $D_i \prec X_i^1$. Finally, we add the sets $A_i \setminus \{r_i\}$, $B_i \setminus \{r_i\}$, $C_i \setminus \{s_i\}$ and $D_i \setminus \{s_i\}$ and enforce $B_i \setminus \{r_i\} \prec D_i \setminus \{s_i\}$ and $C_i \setminus \{s_i\} \prec A_i \setminus \{r_i\}$. We call the sets added in this step the class 4 sets. These enforced preference are shown as solid arrows in Figure 2.

Now, we claim that it is not possible for a weak order $\preceq$ to satisfy dominance and independence with respect to $\leq$ if $X_i^1 \sim X_i^2$ holds. Assume otherwise that $\preceq$ is a weak order that satisfies dominance and independence with respect to $\leq$ such that $X_i^1 \sim X_i^2$ holds. Then $D_i \prec X_i^1 \preceq X_i^2 \prec C_i$ implies $D_i \prec C_i$ by transitivity and hence $D_i \setminus \{s_i\} \preceq C_i \setminus \{s_i\}$ by reverse independence. Similarly, $A_i \prec X_i^2 \preceq X_i^1 \prec B_i$ implies $A_i \prec B_i$ by transitivity and hence $A_i \setminus \{r_i\} \preceq B_i \setminus \{r_i\}$ by reverse independence. However, this leads to a contradiction by

$$A_i \setminus \{r_i\} \preceq B_i \setminus \{r_i\} \prec D_i \setminus \{s_i\} \preceq C_i \setminus \{s_i\} \prec A_i \setminus \{r_i\}$$

Now, it is clear that $\mathcal{X}$ can only be DI-orderable with respect to $\leq$ if ϕ is a positive instance of Π_2-SAT.

It remains to show that the modified family $\mathcal{X}$ is strongly DI-orderable if ϕ is a positive instance of Π_2-SAT. The new sets $\{z_j^e\}, \{z_j^e, \overline{z_j^e}\}, \{\overline{z_j^e}\}$ can be added to the order as described in the proof of Theorem 5. Furthermore, the set $(X_i^a \setminus \{y_j^b, x_i^-\}) \cup \{\max_j^1\}$ can be added in the order $\preceq$ right after $(X_i^a \setminus \{y_j^b\}) \cup \{\max_j^1\}$ if $x_i^- <' v_1$ or right before $(X_i^a \setminus \{y_j^b\}) \cup \{\max_j^1\}$ if $v_1 <' x_i^-$. The other new class 3 sets can be added the same way. It remains to add the class 4 sets to the order. The class 4 sets used to enforce the preferences $A_i \prec X_i^2$, $X_i^1 \prec B_i$, $X_i^2 \prec C_i$ and $D_i \prec X_i^1$ can be ordered the same way as the class 3 sets. As before, we use the fact that we can assume $v_1 <' v_2$ to ensure that this order is compatible with the enforced preferences. For a specific variable V_i we set by construction either $X_i^1 \prec X_i^2$ or $X_i^2 \prec X_i^1$. We assume $X_i^1 \prec X_i^2$. The other case is symmetric. Then we add D_i in $\preceq$ exactly before X_i^1 and B_i exactly after X_i^1. Similarly, we add A_i exactly before X_i^2 and C_i exactly after X_i^2. Then we have

$$D_i \prec X_i^1 \prec B_i \prec A_i \prec X_i^2 \prec C_i$$

which is compatible with the forced preferences.

Now, consider the group $A_i \setminus \{r_i\}, B_i \setminus \{r_i\}, C_i \setminus \{s_i\}$ and $D_i \setminus \{s_i\}$. We observe that all sets in this group differ in at least two elements. Therefore, we only have to set $B_i \setminus \{r_i\} \preceq A_i \setminus \{r_i\}$ and $D_i \setminus \{s_i\} \preceq C_i \setminus \{s_i\}$ in order to satisfies independence. Furthermore, we have to satisfy dominance if r_i and/or s_i are the largest element of the set they are removed from. This can be satisfied by a straightforward construction that respects the enforced preferences unless r_i is the maximal element of A_i and s_i is the minimal element of C_i or alternatively if r_i is the maximal element of B_i and s_i is the minimal element of D_i. We describe the construction for the first case: We have to set $A_i \setminus \{r_i\} \prec A_i$ and

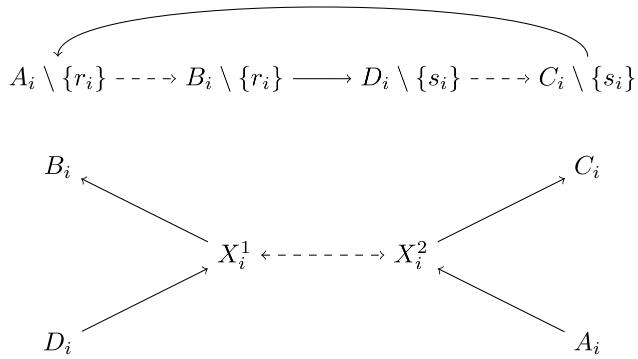

Figure 2: Enforcing strictness.

[5]Every linear order satisfying independence has to satisfy reverse independence, i.e. $A \cup \{x\} \prec B \cup \{x\}$ implies $A \preceq B$: Assume otherwise $B \prec A$ holds, then by independence $B \cup \{x\} \preceq A \cup \{x\}$ must hold, contradicting $A \cup \{x\} \prec B \cup \{x\}$. Hence by the totality of $\preceq$ we have $A \preceq B$.

$C_i \prec C_i \setminus \{s_i\}$ which implies $A_i \setminus \{r_i\} \prec C_i \setminus \{s_i\}$ contrary to the preference we wanted to enforce in the construction. We use the fact that r_i is the maximal element of A_i and s_i is the minimal element of C_i achieve this. Let $z_i^{r,a}$ and $\max_i^{r,a}$ be the new elements used to enforce $\{z_i^{r,a}\} \prec A_i \setminus \{r_i\}$. Then we set A, B for the sets A, B such that $\max_i^{r,a} \in A, B$ if

- $v_2 \in A$ and $v_2 \notin B$
- $v_2 \in A, B$ or $v_2 \notin A, B$ and $\max_{\leq'}(A_L \triangle B_L) \in B$,
- $v_2 \in A, B$ or $v_2 \notin A, B$, $A_L = B_L$ and $\min_{\leq'}(A_S \triangle B_S) \in A$.

where $A_L := \{x \in A \mid \max_i^{r,a} <' x\}$ and $A_S := \{x \in A \mid x <' \max_i^{r,a}\}$. This order satisfies dominance and independence because the element s_i is smaller than v_2 by assumption and removed later in the sequence $A_1, \ldots, A_k$. Furthermore, this implies $(A_i \setminus \{r_i\}) \cup \{\max_i^{r,a}\} \prec \{z_i^{r,a}, \max_i^{r,a}\}$, which implies $A_i \setminus \{r_i\} \prec \{z_i^{r,a}\}$. This allows us to set $A_i \setminus \{r_i\} \prec C_i \setminus \{s_i\}$. Then we can place $A_i \setminus \{r_i\}$ just before A_i and $C_i \setminus \{s_i\}$ just after C_i to get an order that satisfies dominance and independence. □

Proof of Corollary 7

We claim that STRONGER DI-ORDERABILITY would be in coNP if there exists a polynomial time algorithm that produces on input $(X, \mathcal{X}, \leq)$ a weak order $\preceq$ on $\mathcal{X}$ that satisfies dominance and independence. Observe that there exists a linear order $\leq$ on X that can not be lifted if and only if $(X, \mathcal{X})$ is negative instance of STRONGER DI-ORDERABILITY. Hence $\leq$ is a certificate (of polynomial size) for the fact that $(X, \mathcal{X})$ is a negative instance. Furthermore, one can check the certificate by running $\mathbb{A}$ on $(X, \mathcal{X}, \leq)$. Then, one only needs to check that the produced order does not satisfy dominance and strict independence. By definition, this can only be the case if $(X, \mathcal{X})$ is a negative instance of STRONGER DI-ORDERABILITY. The argument for strict independence is analog. □

Proof of Proposition 10

Let ϕ be an instance of TAUT. We assume w.l.o.g. that no variable occurs twice in the same clause. We construct an instance $(S, \mathcal{X})$ of the STRONG PARTIAL DI^S-ORDERABILITY. For every variable X_i in ϕ we add new elements x_i^1 and x_i^2 to S. We call the set of these elements X. We will treat every order on S as encoding a truth assignment by equating $x_i^1 < x_i^2$ to "X_i is true" and $x_i^2 < x_i^1$ to "X_i is false". Furthermore, we add for every clause new variables y_j^1, y_j^2. We call the set of these elements Y. We also add for every clause C_j elements c_j as well as d_j^k and e_j^k for $k \leq 3$. Finally, we add new variables u, v, z_1 and z_2. In the following we call any linear order on S that is derived by replacing X with an arbitrary linear order on the elements in X in the following linear order

$$u < c_1 < \cdots < c_m < y_1^1 < \cdots < y_m^1 < d_1^1 < \cdots < d_m^3 < X < e_1^1 < \cdots < e_m^3 < y_1^2 < \cdots < y_m^2 < z_1 < z_2 < w$$

a critical linear order. In the following, we write $\prec_{\min}$ for the minimal partial order satisfying dominance and strict independence with respect to some linear order on S.

Next, we build the family $\mathcal{X}$. We do this in a way such that $\mathcal{X}$ is not strongly DI^S-orderable if there is a non-satisfying assignment of ϕ. First, we add singletons for all elements of X and Y, and $\{x_i^1, x_i^2\}$ for all elements of X. Then, for every linear order $\leq$ we have $\{x_i^1\} \prec_{\min} \{x_i^1, x_i^2\} \prec_{\min} \{x_i^2\}$ and hence $\{x_i^1\} \prec_{\min} \{x_i^2\}$ if $x_i^1 < x_i^2$ and, on the other hand, $\{x_i^2\} \prec_{\min} \{x_i^1, x_i^2\} \prec_{\min} \{x_i^1\}$ and hence $\{x_i^2\} \prec_{\min} \{x_i^1\}$ if $x_i^2 < x_i^1$

Next, we add sets such that there is a critical linear order $\leq$ on S such that we have $\{y_i^1\} \prec_{\min} \{y_i^2\}$ for all $i \leq m$ if and only if ϕ is not a tautology. For every clause $C_j = X_{i_1} \wedge X_{i_2} \wedge X_{i_3}$ we add sets

$$\{y_j^1, d_j^k\}, \{y_j^1, d_j^k, x_{i_k}^1\}, \{d_j^k, x_{i_k}^1\}$$

for all $k \in \{1, 2, 3\}$ as well as

$$\{x_{i_k}^2, e_j^k\}, \{x_{i_k}^2, e_j^k, z_1\}, \{x_{i_k}^2, e_j^k, z_1, z_2\}, \{e_j^k, z_1, z_2\}, \{e_j^k, z_1, z_2, y_2^j\}, \{z_1, z_2, y_2^j\}, \{z_2, y_2^j\}.$$

If any of the variables occurs negatively in C_j, we switch $x_{i_k}^1$ and $x_{i_k}^2$ in the construction. We claim that these sets ensure that $\{y_j^1\} \prec_{\min} \{y_j^2\}$ holds for any critical linear order whenever at least one literal in C_j is false. We have

$$\{y_j^1\} \prec_{\min} \{y_j^1, d_j^k\} \prec_{\min} \{y_j^1, d_j^k, x_{i_k}^1\} \prec_{\min} \{d_j^k, x_{i_k}^1\} \prec_{\min} \{x_{i_k}^1\}$$

by dominance and hence $\{y_j^1\} \prec_{\min} \{x_{i_k}^1\}$ by transitivity. By a similar argument, we get $\{x_{i_k}^2\} \prec \{y_j^2\}$. Hence, $\{x_{i_k}^1\} \prec_{\min} \{x_{i_k}^2\}$ implies $\{y_j^1\} \prec_{\min} \{y_j^2\}$ by transitivity.

Next we add sets that lead to a contradiction if $\{y_j^1\} \prec_{\min} \{y_j^2\}$ holds for all clauses C_j. First we add

$$\{u\}, \{u, c_1\}, \{u, c_1, y_1^1\}, \{u, c_1, y_1^1, v\}, \{u, v\}.$$

Then we know for any critical linear order that

$$\{u\} \prec_{\min} \{u, c_1\} \prec_{\min} \{u, c_1, y_1^1\}$$

holds by dominance and therefore we have $\{u, v\} \prec_{\min} \{u, c_1, y_1^1, v\}$. Now we add for every clause $\{c_j, y_j^1\}$ and $\{c_j, y_j^2\}$. Then we add new sets that are constructed by incrementally adding to both sets, one by one, first all elements c_{j-1} to c_1, then all elements y_{j-1}^2 to y_1^2 and finally u and v in that order. In other words we add

$$\{c_{j-1}, c_j, y_j^1\} \text{ and } \{c_{j-1}, c_j, y_j^2\}, \{c_{j-2}, c_{j-1}, c_j, y_j^1\} \text{ and } \{c_{j-2}, c_{j-1}, c_j, y_j^2\}, \ldots, \{c_1, \ldots, c_j, y_j^1\} \text{ and } \{c_1, \ldots, c_j, y_j^2\}$$

as well as

$$\{c_1, \ldots, c_j, y_j^1, y_{j-1}^2\} \text{ and } \{c_1, \ldots, c_j, y_j^2, y_{j-1}^2\}, \ldots, \{c_1, \ldots, c_j, y_j^1, \ldots, y_{j-1}^2\} \text{ and } \{c_1, \ldots, c_j, y_j^2, \ldots, y_1^2\}$$

and finally

$$\{u, c_1, \ldots, c_j, y_j^1, \ldots, y_{j-1}^2\} \text{ and } \{u, c_1, \ldots, c_j, y_j^2, \ldots, y_1^2\},$$

as well as

$$\{u, c_1, \ldots, c_j, y_j^1, \ldots, y_{j-1}^2, v\} \text{ and } \{u, c_1, \ldots, c_j, y_j^2, \ldots, y_1^2, v\}.$$

By construction

$$\{u, c_1, \ldots, c_j, y_j^1, \ldots, y_{j-1}^2, v\} \prec_{\min} \{u, c_1, \ldots, c_j, y_j^2, \ldots, y_1^2, v\}$$

holds for the minimal partial order satisfying dominance and strict independence for any linear order on X if and only if $\{y_j^1\} \prec_{\min} \{y_j^2\}$ holds for that partial order.

Next we add $\{u, c_1, \ldots, c_j\}$, $\{u, c_1, \ldots, c_{j+1}\}$ and $\{u, c_1, \ldots, c_{j+1}, y_{j+1}^1\}$. Then we add new sets derived as above by adding to both sets first all elements y_j^2 to y_1^2 and then v, one by one, in that order until we reach

$$\{u, c_1, \ldots, c_j, y_j^2, \ldots, y_1^2, v\} \text{ and } \{u, c_1, \ldots, c_{j+1}, y_{j+1}^1, y_j^2, \ldots, y_1^2, v\}.$$

Then

$$\{u, c_1, \ldots, c_j, y_j^2, \ldots, y_1^2, v\} \prec_{\min} \{u, c_1, \ldots, c_{j+1}, y_{j+1}^1, y_j^2, \ldots, y_1^2, v\}$$

holds for the critical linear order by strict independence because

$$\{u, c_1, \ldots, c_j\} \prec \{u, c_1, \ldots, c_{j+1}\} \prec \{u, c_1, \ldots, c_{j+1}, y_{j+1}^1\}$$

holds by dominance. Finally, we add $\{v\}$ and then $\{y_1^2, v\}$, $\{y_2^2, y_1^2, v\}$ and so on till we reach

$$\{c_1, \ldots, c_m, y_m^2, \ldots, y_1^2, v\}.$$

This forces for any critical linear order

$$\{u, c_1, \ldots, c_m, y_m^2, \ldots, y_1^2, v\} \prec_{\min} \{u, v\}.$$

Now, by construction and transitive we have for any critical linear order

$$\{u, v\} \prec_{\min} \{u, c_1, y_1^1, v\} \prec_{\min} \{u, c_1, y_1^2, v\} \prec_{\min} \{u, c_1, c_2, y_2^1, y_1^2, v\} \prec_{\min} \cdots \prec_{\min} \{u, c_1, \ldots, c_m, y_m^2, \ldots, y_1^2, v\} \prec_{\min} \{u, v\}$$

if (and only if) $\{y_j^1\} \prec_{\min} \{y_j^2\}$ holds for all clauses, i.e. if the critical linear order codes an unsatisfying assignment. It follows that if ϕ is not a tautology, then $(X, \mathcal{X})$ is not strongly partial DI^S-orderable.

It remains to show that $(X, \mathcal{X})$ is strongly partial DI^S-orderable if ϕ is a tautology. Let $\leq$ be a linear order on S. We construct a partial order $\preceq$ that satisfies dominance and strict independence with respect to $\leq$. To avoid unnecessary case distinctions, we will describe the construction only for clauses with all positive variables. The only change in construction required for negative variables is switching x_i^1 and x_i^2. By Lemma 16, we can assume w.l.o.g. that $z_1 < z_2$. First we add the forced preferences between $\{x_i^1\}, \{x_i^1, x_i^2\}$ and $\{x_i^2\}$. Next, we consider the sets containing an element d_j^k. We add all preferences that are implied by dominance between sets from

$$\{y_j^1\}, \{y_j^1, d_j^k\}, \{y_j^1, d_j^k, x_{i_k}^1\}, \{d_j^k, x_{i_k}^1\}, \{x_{i_k}^1\}$$

and close under transitivity. The only possible application of strict independence on these sets is that any preference between $\{y_j^1\}$ and $\{x_{i_k}^1\}$ has to be lifted to $\{y_j^1, d_j^k\}$ and $\{d_j^k, x_{i_k}^1\}$. By construction however, there can only be a preference between $\{y_j^1\}$ and $\{x_{i_k}^1\}$ forced by dominance and transitivity if the same preference holds between $\{y_j^1, d_j^k\}$ and $\{d_j^k, x_{i_k}^1\}$. Because we assume that no variable occurs twice in a clause, a preference between $\{y_j^1\}$ and $\{x_{i_k}^1\}$ can not later be introduced through sets containing another $d_{j'}^{k'}$. Finally, to satisfy dominance and transitivity we have to add for all x_i^1 the preference $\{x_i^1, d_{j_1}^{k_1}\} \prec \{x_i^1, d_{j_2}^{k_2}\}$ for all $d_{j_1}^{k_1}, d_{j_2}^{k_2}$ such that $d_{j_1}^{k_1} < d_{j_2}^{k_2}$ holds.

Using a similar construction, we can order all sets containing an element e_j^k if we replace x_i^1 by x_i^2 and y_j^1 by $\{z_1, z_2, y_j^2\}$. Finally, we add the enforced preference between $\{z_2, y_j^2\}$ and $\{y_j^2\}$ as well as $\{z_1, z_2, y_j^2\} \prec \{z_2, y_j^2\}$. The later is enforced by dominance as we assume $z_1 < z_2$. Then we close everything under transitivity. By construction, this does not produce any new instances of strict independence.

Now, we observe that $\{z_1, z_2, y_j^2\} \prec \{z_2, y_j^2\}$ implies that $\{y_j^1\} \prec \{y_j^2\}$ can only hold if $\{x_i^1\} \prec \{x_i^2\}$ holds for a variable occurring in clause C_j, i.e. if clause C_j is not satisfied. As ϕ is a tautology, there is clause C_l that is satisfied by the assignment coded by $\leq$. Hence, $\{y_l^1\} \prec \{y_l^2\}$ does not hold. We now consider the sets containing an element c_i for some i. We partition these sets in partitions $P_1, \ldots, P_m$ based on the largest i for which they contain c_i. We set $S_1 \prec S_2$ if $S_1 \in P_{i_1}$, $S_2 \in P_{i_2}$ and one of the following holds:

- $c_{i_1} < c_{i_2}$ and $i_1, i_2 < l$
- $c_{i_1} < c_{i_2}$ and $l < i_1, i_2$

Then any set that contains c_i also contains y_i except $\{u, c_1, \ldots, c_i\}$. Hence the only possible application of dominance between sets of different partitions is $\{u, c_1, \ldots, c_i\} \prec \{u, c_1, \ldots, c_{i+1}\}$ which is satisfied by construction for $i, i+1 \neq l$. Now for any set in any partition P_i such that $i \neq l$ we set $S \prec S'$ if $y_i^1 \in S$ and $y_i^1 \notin S'$. This covers all applications of strict independence in a partition. Finally, we add all preferences that are forced by dominance in a partition and close under transitivity. We observe that $S, S \cup \{x\} \in P_i$ implies either $y_i^1 \in S, S \cup \{x\}$ or $y_i^1 \notin S, S \cup \{x\}$, hence this can not lead to a contradiction. Now, for a set S in P_l such that $y_l^1 \in S$ we set

- $S' \prec S$ if $S' \in P_i$ for $i < l$ and $c_i < c_l$

- $S \prec S'$ if $S' \in P_i$ for $i < l$ and $c_l < c_i$

Furthermore, for a set S in P_l such that $y_l^1 \notin S$ we set

- $S' \prec S$ if $S' \in P_i$ for $l < i$ and $c_i < c_l$
- $S \prec S'$ if $S' \in P_i$ for $l < i$ and $c_l < c_i$

And finally, we add again all preferences forced by dominance and close by transitivity. As $\{y_l^1\}$ and $\{y_l^2\}$ are incomparable in $\preceq$ this order is consistent. Furthermore, $\{u, c_1, y_1^1, z_1, z_2, v\}$ and $\{u, c_1, \ldots, c_m, y_m^2, \ldots, y_1^2, v\}$ are incomparable in $\preceq$. This allows us to add all preferences forced by dominance and strict independence regarding $\{u\}, \{v\}$ and $\{u, v\}$ without creating a contradiction. By construction, $\preceq$ is now a partial order that satisfies dominance and strict independence. □

Proof of Theorem 13

SUCCINCT DI^S-ORDERABILITY can be solved in NEXP-time by explicitly computing the family $\mathcal{X}$ and then solving the (exponentially larger) explicit problem in NP-time.

For the hardness, we only have to check that the presented reduction is computable in polylog-time. Then, by the Conversion Lemma, there is a ptime reduction from SUCCINCT SAT to SUCCINCT DI^S-ORDERABILITY resp. SUCCINCT STRONG DI^S-ORDERABILITY. The NEXP-hardness of both problems then follows as SUCCINCT SAT is known to be NEXP-complete (Papadimitriou 1994). We have to show that we can compute a single bit of the output in polylog-time if we have random access to the input. For this, we have to take the binary representation of the SAT into account. Unfortunately, (Papadimitriou 1994) does not specify a binary representation for the NEXP-hardness proof. However, the proof given in the book is not sensitive to the representation as long as it is reasonable. The same is true for our proof. Reasonable means in our context that it is possible to determine the number of variables n and clauses m in polylog-time. For any sensible encoding of 3-CNF this is either explicitly encoded or can be determined via binary search. Furthermore, we assume that one only needs polylog-time to read the i-th variable in the j-th clause. This is trivially true if we assume that every clause is encode by the same amount of bits. It is easy to see that the proof in (Papadimitriou 1994) of the NEXP-hardness of SUCCINCT SAT works for such an encoding.

Now, we fix a binary representation for instances of DI^S-ORDERABILITY resp. STRONG DI^S-ORDERABILITY. First, we encode the number of elements k of X in binary. Then, the family $\mathcal{X}$ is encoded as a series of strings of length k, where a 1 in position l means the l-th element of X is in the set and a 0 in position l means the l-th element is not in the set. For an instance of DI^S-ORDERABILITY, the linear order $\leq$ is given by the natural order on these positions.

First observe that the size of X is $4n + 12m + 3$ and the size of $\mathcal{X}$ is $p(n, m)$ for some polynomial $p(x, y)$. Therefore, we can determine it in polylog-time. Now, assume we want to decide whether the i-th bit of the output is 0 or 1. It is clear that this can be done in polylog-time if the i-th bit is part of the representation of the size of X. Assume that the i-th bit determines if the l-th element x is part of a k-th set A. We can assume that we fixed an order in which we generate the sets in $\mathcal{X}$ such that we can compute from m, n and i which set A is supposed to be. Observe that if x is not of the form x_j^+ or x_j^- then, this already suffices to decide whether x is in A. On the other hand, if $x = x_{j,a}^+$ or $x = x_{j,a}^-$ and A is a class 1 set, then this already suffices. Finally, if $x = x_{j,a}^+$ or $x = x_{j,a}^-$ and A is not a class 1 set then the question whether x is in A only depends on the question if X_j occurs (positively or negatively) in that clause in the right position.

The properties of the reduction from SAT to STRONG DI^S-ORDERABILITY resp. DI^S-ORDERABILITY used in the proof above hold also for the reduction from SAT to STRONG DI-ORDERABILITY resp. DI-ORDERABILITY. Therefore, Theorem 14 can be proven using the same argumentation as above. □

Proof of Theorem 14

We observe that the reduction from TAUT to STRONG PARTIAL DI^S-ORDERABILITY satisfies the same properties as the reduction from SAT to STRONG DI^S-ORDERABILITY, i.e. the number of elements in S as well as the number and size of the sets in $\mathcal{X}$ only depends on the size and not on the structure of the formula ϕ. Furthermore, if a element is in a set or not only depends on one specific clause. Therefore, the reduction can be done in polylog-time. By the Conversion Lemma and the coNEXP-completeness of TAUT, this suffices to show that STRONG PARTIAL DI^S-ORDERABILITY is coNEXP-complete. □