Proportional Belief Merging*

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Abstract

In this paper we introduce proportionality to belief merging. Belief merging is a framework for aggregating information presented in the form of propositional formulas, and it generalizes many aggregation models in social choice. In our analysis, two incompatible notions of proportionality emerge: one similar to standard notions of proportionality in social choice, the other more in tune with the logic-based merging setting. Since established merging operators meet neither of these proportionality requirements, we design new proportional belief merging operators. We analyze the proposed operators against established rationality postulates, finding that current approaches to proportionality from the field of social choice are, at their core, incompatible with standard rationality postulates in belief merging. We provide characterization results that explain the underlying conflict, and provide a complexity analysis of our novel operators.

1 Introduction

Proportionality is one of the central fairness notions studied in social choice theory [4, 6, 25], arising whenever a collective decision should reflect the amount of support in favor of a set of issues. Thus, notions of proportionality are key when it is desirable that preferences of larger groups have more influence on the outcome, while preferences of smaller groups are not neglected.

The idea of proportional representation shows up in many application scenarios: it is a key ingredient of parliamentary elections [3] and, more generally, of multiwinner voting, i.e., the task of electing a committee of multiple candidates [13]. Recent work has set out to extend the notion of proportionality from mathematically simple formalisms (mainly the apportionment setting) to more general settings, with significant progress in areas such as approval-based multiwinner voting [1, 28], ordinal multiwinner voting [8, 9], proportional rankings [29], and multi-attribute committees [23].

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In this paper we introduce proportionality to the very general framework of *belief merging* [20, 17, 18], in which agents can combine their individual positions on a set of issues in order to obtain a collective solution, with the added option of imposing constraints on admissible outcomes. Though the agents' individual positions are called *beliefs*, the belief merging framework is versatile enough that it can accommodate a broad range of attitudes (e.g., beliefs, preferences, judgments, goals or items of knowledge), as long as these, together with the constraint and the outcome, can be expressed as formulas in a logical language. The key challenges of such a process are that agents may hold mutually conflicting beliefs, and that beliefs may reflect complex interdependencies between issues. The theory of belief merging then offers (i) a range of methods, called *belief merging operators*, for aggregating beliefs and (ii) postulates used to assess the rationality of the operators.

The most prominent belief merging operators studied so far tend to fall into two main categories: operators following the majority opinion, and which can be said to embody a *utilitarian* stance; and operators that place particular emphasis on the worst-off agents, and which can be said to be based on an *egalitarian* viewpoint. Our aim is to find a compromise between these two opposing positions, which, in a belief merging scenario, translates as the following desideratum: if a large enough proportion of the agents share common beliefs, then these beliefs should be reflected at the collective level, to a degree matching their proportion. Despite its intuitive appeal, such a proportionality requirement has yet to find its way in the study of belief merging operators.

In defining proportional belief merging operators we rely on the Proportional Approval Voting (PAV) rule, studied in multiwinner voting scenarios and known to satisfy particularly strong proportionality requirements [1]. Based on PAV, we introduce three belief merging operators: the PAV operator, the bounded PAV operator and the harmonic Hamming operator. All these operators fall into the class of *satisfaction-based operators*, introduced by us (Section 4) as an alternative to the standard way of representing merging operators, which is distance-based.

We look at the proposed belief merging operators from three perspectives. Firstly, in Section 5, the operators are placed against the standard belief merging IC-postulates. We show that any belief merging operator directly extending PAV cannot be compatible with all IC-postulates; in particular, such an operator will not satisfy postulate IC_2 which stipulates that any admissible agreement among agents shall be part of the merged result. We also provide a characterization of operators that fail IC_2 based on properties of the ranking a satisfaction-based operator induces, which provides an alternative view on why the PAV approach to proportionality is inconsistent together with IC_2. However, we show that the bounded PAV operator can be characterized as the only merging operator (of a certain natural class) that extends PAV and satisfies all other postulates. While the harmonic Hamming operator is defined via the harmonic sum used by PAV, it does not generalize PAV. Thus, the aforementioned impossibility does not hold; indeed, the harmonic Hamming operator satisfies all standard IC postulates IC_{0-8}.

Secondly, in Section 6, we introduce two basic proportionality postulates for the belief merging domain. The first one (*classical proportionality*) is the kind of proportionality requirement typically studied in social choice settings, in particular in the apportionment setting [3]. This notion is based on the assumption that agents derive utility from positive occurrences, i.e., from approved candidates being selected in the collective choice. The second notion (*binary proportionality*) is closer to the logical nature of belief merging. Here, no difference is made between positive and negative agreement: the agents' utility derives from the (Hamming) distance between their preferences and the collective choice. We show that these two notions are mutually exclusive and contradict each other. Furthermore, we show by example that established belief merging operators satisfy neither of these two postulates. In contrast, the aforementioned PAV and bounded PAV operators satisfy classical proportionality and the harmonic Hamming operator satisfies binary proportionality.

Thirdly, in Section 7, we study the complexity of our proposed merging operators. Our results are that our novel operators fall into similar complexity classes as established merging operators, which shows that the introduction of proportionality comes at a moderate computational cost.

As mentioned before, belief merging can be seen as a general framework. In Section 3, we make this argument precise for approval-based committee elections, and we show that our work has implications for other settings (Sect. 8). In particular, our work yields new proportional goal-based voting rules [26] and approval-based multiwinner rules with a variable number of winners [16, 14], and gives insights for proportional judgment aggregation [24, 11, 12].

Proofs of formal results are detailed in the supplement.

2 Belief Merging

We assume a set $\mathcal{A}$ of m propositional atoms, with $\mathcal{L}$ the set of propositional formulas generated from $\mathcal{A}$ using the usual connectives. An *interpretation* w is a truth-value assignment to atoms in $\mathcal{A}$, and we denote by $\mathcal{U}$ the set of all interpretations over the set $\mathcal{A}$ of atoms. We typically write interpretations as words where letters are the atoms assigned to true. If v and w are interpretations, the *symmetric difference* $v \triangle w$ *between* v *and* w is defined as $v \triangle w = (v \setminus w) \cup (w \setminus v)$. The *Hamming* and *drastic* distances d_{H} and d_{D}, respectively, are defined as $d_{\mathrm{H}}(v, w) = |v \triangle w|$ and $d_{\mathrm{D}}(v, w) = 0$, if $v = w$, and 1 otherwise. If $\varphi \in \mathcal{L}$ is a propositional formula and w is an interpretation, w is a *model of* φ if w satisfies φ. We write $[\varphi]$ for the set of models of φ. If $\varphi_1, \varphi_2 \in \mathcal{L}$, we say that $\varphi_1 \models \varphi_2$ if $[\varphi_1] \subseteq [\varphi_2]$, and that $\varphi_1 \equiv \varphi_2$ if $[\varphi_1] = [\varphi_2]$. A formula φ is *consistent*, or *satisfiable*, if $[\varphi] \neq \emptyset$.

A *propositional profile* $P = (\varphi_1, \ldots, \varphi_n)$ is a finite tuple of consistent propositional formulas. We will assume that each formula φ_i in a profile P corresponds to an agent i. If P_1 and P_2 are profiles, we write $P_1 + P_2$ for the profile obtained by appending P_2 to P_1. If φ_i is a formula and there is no danger of ambiguity, we write $P + \varphi_i$ instead of $P + (\varphi_i)$. A *merging operator* Δ is a function mapping a profile P of consistent formulas and a propositional formula μ, called *the constraint*, to a propositional formula, written $\Delta_\mu(P)$. Two merging operators Δ^1 and Δ^2 are *equivalent* if $\Delta^1_\mu(P) \equiv \Delta^2_\mu(P)$, for any profile P and constraint μ. The following postulates are typically taken to provide a core set of rationality constraints that any merging operator Δ is expected to satisfy [19, 18]:

(IC_0) $\Delta_\mu(P) \models \mu$.

(IC_1) If μ is consistent, then $\Delta_\mu(P)$ is consistent.

(IC_2) If $\bigwedge P \wedge \mu$ is consistent, then $\Delta_\mu(P) \equiv \bigwedge P \wedge \mu$.

(IC_3) If $P_1 \equiv P_2$ and $\mu_1 \equiv \mu_2$, then $\Delta_{\mu_1}(P_1) \equiv \Delta_{\mu_2}(P_2)$.

(IC_4) If $\varphi_1 \models \mu$ and $\varphi_2 \models \mu$, then $\Delta_\mu(\varphi_1, \varphi_2) \wedge \varphi_1$ is consistent if and only if $\Delta_\mu(\varphi_1, \varphi_2) \wedge \varphi_2$ is consistent.

(IC_5) $\Delta_\mu(P_1) \wedge \Delta_\mu(P_2) \models \Delta_\mu(P_1 + P_2)$.

(IC_6) If $\Delta_\mu(P_1) \wedge \Delta_\mu(P_2)$ is consistent, then $\Delta_\mu(P_1 + P_2) \models \Delta_\mu(P_1) \wedge \Delta_\mu(P_2)$.

(IC_7) $\Delta_{\mu_1}(P) \wedge \mu_2 \models \Delta_{\mu_1 \wedge \mu_2}(P)$.

(IC_8) If $\Delta_{\mu_1}(P) \wedge \mu_2$ is consistent, then $\Delta_{\mu_1 \wedge \mu_2}(P) \models \Delta_{\mu_1}(P) \wedge \mu_2$.

These postulates are best understood as axiomatizing a decision procedure based on the aggregation of information coming from different sources (the formulas in P), under a constraint μ that must be satisfied by the result (postulate IC_0). The result should be consistent (postulate IC_1), independent of the syntax of the formulas involved (postulate IC_3), include outcomes that are unanimously accepted across subprofiles (postulates IC_{5-6}) and coherent when varying the constraint (postulates IC_{7-8}). Additionally, postulate IC_2 requires that if there is any agreement between the formulas in P and μ, then the merged result is nothing more than the agreed upon outcomes; and postulate IC_4 stipulates that merging two formulas φ_1 and φ_2 should be fair, in the sense that if the result contains outcomes consistent with one of the formulas, it should contain results consistent with the other as well. We will see that the latter two postulates are problematic for proportionality-driven merging operators.

Standard ways of constructing merging operators that satisfy postulates IC_{0-8} are based on the idea of minimizing overall distance to the profile $P = (\varphi_1, \ldots, \varphi_n)$, and rely on two parameters [17, 18]. The first is a notion of *pseudo-distance* $d\colon \mathcal{U} \times \mathcal{U} \to \mathbb{R}_{\geq 0}$ between interpretations, typically either Hamming distance d_{H} or drastic distance d_{D}. The distance $d(\varphi, w)$ from a formula φ to an interpretation w is then defined as $d(\varphi, w) = \min_{v \in [\varphi]} d(v, w)$. The collective distance w.r.t. profile P is obtained using the second ingredient, an *aggregation function* $f\colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ that, for any integer n, maps a vector of n real numbers to a real number, and is defined as $d^f(P, w) = f(d(\varphi_1, w), \ldots, d(\varphi_n, w))$. Typical aggregation functions are the sum Σ and gmax. By $f = \text{gmax}$ vectors are ordered in descending order. For this aggregation function, the resulting ordered vectors are compared according to a lexicographic order. The *distance-based merging operator* $\Delta^{d,f}$ is defined, for any profile P and formula μ, as a formula $\Delta^{d,f}_\mu(P)$ such that $[\Delta^{d,f}_\mu(P)] = \text{argmin}_{w \in [\mu]} d^f(P, w)$, i.e., as a formula

Table 1: Hamming distances for $\Delta^{\mathrm{H},\Sigma}$ and $\Delta^{\mathrm{H},\mathrm{gmax}}$.

d_{H}	$x_1x_2x_3x_4$	$x_1x_2x_3x_4$	$x_1x_2x_3x_4$	$y_1y_2y_3y_4$	Σ	gmax
$x_1x_2x_3x_4$	0	0	0	8	**8**	$(8, 0, 0, 0)$
$x_1x_2x_3y_1$	2	2	2	6	12	$(6, 2, 2, 2)$
$x_1x_2y_1y_2$	4	4	4	4	16	$(\mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4})$
$x_1y_1y_2y_3$	6	6	6	2	20	$(6, 6, 6, 2)$
$y_1y_2y_3y_4$	8	8	8	0	24	$(8, 8, 8, 0)$

whose models are the models of μ at minimal collective distance to P. When $d = d_{\mathrm{D}}$, the operators $\Delta^{\mathrm{D},\Sigma}$ and $\Delta^{\mathrm{D},\mathrm{gmax}}$ are equivalent and we will heretofore refer to them as Δ^{D}. Thus, we recall three main distance-based operators ($\Delta^{\mathrm{H},\Sigma}$, $\Delta^{\mathrm{H},\mathrm{gmax}}$ and Δ^{D}), all of which are known to satisfy postulates IC_{0-8} [19, 18].

Example 1. *For the set of atoms $\mathcal{A} = X \cup Y$, where $X = \{x_1, \ldots, x_4\}$ and $Y = \{y_1, \ldots, y_4\}$, take a profile $P = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ with $\varphi_i = (x_1 \wedge x_2 \wedge x_3 \wedge x_4) \wedge (\neg y_1 \wedge \neg y_2 \wedge \neg y_3 \wedge \neg y_4)$, for $i \in \{1, 2, 3\}$, $\varphi_4 = (\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4) \wedge (y_1 \wedge y_2 \wedge y_3 \wedge y_4)$. We obtain that $[\varphi_i] = \{x_1x_2x_3x_4\}$, for $i \in \{1, 2, 3\}$ and $[\varphi_4] = \{y_1y_2y_3y_4\}$. Additionally, take a constraint μ such that $[\mu] = \{x_1x_2x_3x_4,\ x_1x_2x_3y_1,\ x_1x_2y_1y_2,\ x_1y_1y_2y_3,\ y_1y_2y_3y_4\}$. Table 1 displays Hamming distances between models of μ and formulas in P as well as the aggregated distances, for the Σ and* gmax *aggregation functions. We have $d_{\mathrm{H}}^{\Sigma}(P, x_1x_2x_3x_4) < d_{\mathrm{H}}^{\Sigma}(P, x_1x_2x_3y_1)$ and $d_{\mathrm{H}}^{\mathrm{gmax}}(P, x_1x_2y_1y_2) < d_{\mathrm{H}}^{\mathrm{gmax}}(P, x_1x_2x_3y_1)$, since the overall distance $(4, 4, 4, 4)$ lexicographically dominates $(6, 2, 2, 2)$. Optimal outcomes are written in bold, i.e., $[\Delta_{\mu}^{\mathrm{H},\Sigma}(P)] = \{x_1x_2x_3x_4\}$ and $[\Delta_{\mu}^{\mathrm{H},\mathrm{gmax}}(P)] = \{x_1x_2y_1y_2\}$. We also obtain that $[\Delta_{\mu}^{\mathrm{D}}(P)] = \{x_1x_2x_3x_4\}$.*

Example 1 illustrates a general feature of the standard merging operators: $\Delta^{\mathrm{H},\Sigma}$ sees optimal outcomes in utilitarian terms and thereby favors the majority opinion, while $\Delta^{\mathrm{H},\Sigma}$ attempts to improve the standing of the worse off agent, thereby favoring an egalitarian outcome. While such approaches may produce, on occasion, proportional outcomes, they are in no way guaranteed to do so in general.

3 Approval-Based Committee Elections as Instances of Belief Merging

Notions of proportionality have been systematically studied in the social choice literature, notably in the case of Approval-Based Committee (ABC) elections [13]. An ABC election consists of a set of candidates C, a desired size of the committee k, and a preference profile $A = (A_1, \ldots, A_n)$. The preference profile A contains approval ballots, i.e., $A_i \subseteq C$ is the set of candidates agent i approves of. An ABC voting rule outputs one or more size-k subsets of C, the chosen committee(s). The ABC voting rule of interest to us is called

Table 2: PAV scores for a selection of committees of size 4.

PAV	$x_1x_2x_3x_4$	$x_1x_2x_3x_4$	$x_1x_2x_3x_4$	$y_1y_2y_3y_4$	Σ
$x_1x_2x_3x_4$	$h(4)$	$h(4)$	$h(4)$	$h(0)$	6.25
$x_1x_2x_3y_1$	$h(3)$	$h(3)$	$h(3)$	$h(1)$	**6.5**
$x_1x_2y_1y_2$	$h(2)$	$h(2)$	$h(2)$	$h(2)$	6
$x_1y_2y_2y_3$	$h(1)$	$h(1)$	$h(1)$	$h(3)$	4.83
$y_1y_2y_3y_4$	$h(0)$	$h(0)$	$h(0)$	$h(4)$	2.08

*Proportional Approval Voting (*PAV*)* [30]. It is based on the *harmonic function* $h\colon \mathbb{N} \to \mathbb{R}$, defined as $h(\ell) = \sum_{i=1}^{\ell} \frac{1}{i}$ with the added convention that $h(0) = 0$. Given a committee w of size k, the PAV*-score of w w.r.t. A* is $\mathrm{PAV}(A, w) = \sum_{i=1}^{n} h(|A_i \cap w|)$, where h is the harmonic function. The PAV rule applied to the preference profile A, for a desired size k of the committee, is defined as $\mathrm{PAV}_k(A) = \mathrm{argmax}_{w \subseteq C, |w|=k} \mathrm{PAV}(A, w)$, i.e., it outputs committees of size k that maximize the PAV score w.r.t. A.

Example 2. *Take a set $C = X \cup Y$ of candidates, where X and Y are as in Example 1, and a preference profile $A = (A_1, A_2, A_3, A_4)$ with $A_i = [\varphi_i]$, where φ_i are, again, chosen from Example 1. Suppose $k = 4$, i.e., the task is to choose committees of size* 4. *Intuitively, a proportional outcome would consist of three candidates from X and one from Y, to reflect the fact that supporters X outnumber supporters of Y in A by a ratio of* $3 : 1$. *Indeed, this is exactly the type of outcome the* PAV *rule will select. In Table 2, depicting the* PAV *scores of a representative sample of possible winning committees, a committee maximizing the overall* PAV *score w.r.t. A is $x_1x_2x_3y_1$.*

In Example 2 we have identified models of propositional formulas with sets of approved candidates in an ABC election. Indeed, we may pursue this analogy further and show that any ABC election can be rephrased as a belief merging instance. Given an instance of an ABC election, we associate to C the set of propositional atoms $\mathcal{A}_C = C$. To agent i's approval ballot $A_i \subseteq C$ we associate the propositional formula: $\varphi_{A_i} = \bigwedge_{x \in A_i} x \wedge \bigwedge_{x \in C \setminus A_i} \neg x$, the sole model of which is exactly A_i. To the preference profile A we associate the propositional profile $P_A = (\varphi_{A_1}, \ldots, \varphi_{A_n})$. To obtain solutions that adhere to the cardinality constraint k, we choose μ_k to be a formula whose models are all subsets of $\mathcal{A}_C$ of size k. By postulates IC_0 and IC_1, $[\Delta_{\mu_k}(P_A)]$ consists of a non-empty set of models of size k, which can be seen as the winning committees in the ABC election.

In general, any ABC election for size-k committees can be seen as a belief merging instance where the profile consists of formulas with exactly one model and the constraint μ has models of fixed size k. A merging operator Δ *extends* PAV if for all preference profiles A, it holds that $\mathrm{PAV}_k(A) = [\Delta_{\mu_k}(P_A)]$, i.e., the output of the PAV voting rule is the set of interpretations, or sets of atoms, returned by $\Delta_{\mu_k}(P_A)$. In the following we will introduce merging operators that

$$
\begin{array}{r}
\text{approval-based}\left\{\begin{array}{rcl}
s_{\mathrm{AV}}(v,w) & = & |v \cap w| \\
s_{\mathrm{PAV}}(v,w) & = & h(|v \cap w|) \\
s_{\mathrm{bPAV}}(v,w) & = & 2h(|v \cap w|) - h(|w|)
\end{array}\right. \\
\text{binary sat.-based}\left\{\begin{array}{rcl}
s_{\mathrm{hH}}(v,w) & = & h(m - d_{\mathrm{H}}(v,w)) \\
s_{\mathrm{hD}}(v,w) & = & h(m - d_{\mathrm{D}}(v,w))
\end{array}\right.
\end{array}
$$

Figure 1: Proposed satisfaction measures.

extend PAV and another that is inspired by it.

4 Satisfaction-based Merging Operators

The framework of ABC elections presented in Section 3 can be used as a springboard for designing proportional belief merging operators. By conceiving ways in which an agent derives utility from a possible outcome, it becomes possible to reason about the social welfare of merging, i.e., the utility of the agents' society as a whole.

The key notion in doing so is a *satisfaction measure* $s\colon \mathcal{U} \times \mathcal{U} \to \mathbb{R}$, quantifying the amount of satisfaction $s(v,w)$ of interpretation v with interpretation w. The satisfaction $s(\varphi,w)$ of a formula φ with w is then defined as $s(\varphi,w) = \max_{v\in[\varphi]} s(v,w)$. Finally, the *collective satisfaction* $s(P,w)$ *of a profile* P *with* w is defined as $s(P,w) = \sum_{\varphi\in P} s(\varphi,w)$. The *satisfaction-based merging operator* Δ^s outputs a formula $\Delta^s_\mu(P)$ such that $[\Delta^s_\mu(P)] = \mathrm{argmax}_{w\in[\mu]} s(P,w)$, i.e., a formula whose models are exactly the models of μ that maximize satisfaction of P.

Note that we can convert a distance-based merging operator $\Delta^{d,\Sigma}$ (see Section 2) into an equivalent satisfaction-based operator by inverting the distance measure d, i.e., by defining a satisfaction measure s as $s(v,w) = m - d(v,w)$, for any interpretations v and w (remember that m is the number of atoms in $\mathcal{A}$). The resulting satisfaction-based operator is s.t. $\Delta^s_\mu(P) \equiv \Delta^{d,\Sigma}_\mu(P)$, for any profile P and μ. Note that since d is assumed to be symmetric (i.e., $d(v,w) = d(w,v)$, for any interpretations v and w), the satisfaction measure s defined on the basis of it is also symmetric. This being said, in the general case we do not require satisfaction measures to be symmetric, and hence satisfaction-based operators as defined here form a more general class than distance-based operators $\Delta^{d,\Sigma}$, where d is a pseudo-distance.

The concrete satisfaction measures we propose are defined, for any interpretations v and w, in Figure 1, and generate two groups of operators. The *approval-based* operators, consisting of the AV *operator* Δ^{AV}, the PAV *operator* Δ^{PAV} and the *bounded* PAV *operator* Δ^{bPAV}, mimic the behavior of an ABC voting rule (see Section 3) in that they compute satisfaction of v with w based on how many atoms v and w have in common, similarly to how satisfaction of an approval ballot A_i with a potential committee w is based on how many approved candidates in A_i find themselves in w. Note that, while an ABC voting rule is defined only for committees of fixed size, the merging operators we propose select among interpretations of any size. Nonetheless, it is straightforward to see that if the allowed outcomes (here, models of the constraint μ) are restricted to a given size, then the operators Δ^{PAV} and Δ^{bPAV} are equivalent and extend, in the sense described in Section 3, the PAV voting rule.

The operator Δ^{AV} is put forward as a benchmark approval-based operator, based on a satisfaction measure that simply counts the atoms v and w have in common: in particular, Δ^{AV} does not incorporate any proportionality ideas. Consequently the Δ^{AV} operator does not extend PAV, and, as shown in Section 6, it does not meet any of the proportionality requirements we propose. The Δ^{PAV} operator refines Δ^{AV} by using the harmonic function h, known to behave well w.r.t. proportionality requirements [1], in order to compute satisfaction. Intuitively, the harmonic function reflects the "diminishing returns" of added satisfaction: the difference between $h(x)$ and $h(x+1)$ gets smaller as x increases. Thus, the operator Δ^{PAV} is a prime candidate for a proportional satisfaction-based merging operator. Nonetheless, Δ^{PAV} has several shortcomings, which serve as motivation for the remaining operators.

One drawback of Δ^{PAV} is that it favors larger interpretations if available (Example 3), i.e., it tries to increase agents' satisfaction by setting as many atoms to true as possible. Such an inflationary strategy may be undesirable in practice and, in a belief merging setting, interferes with postulate IC_4.

Example 3. *For $\mathcal{A} = \{x_1, x_2\}$, profile $P = (\varphi_1, \varphi_2)$, with $[\varphi_1] = \{x_1\}$ and $[\varphi_2] = \{x_1x_2\}$, and constraint μ such that $[\mu] = \{x_1, x_1x_2\}$, we obtain that $[\Delta^{\mathrm{PAV}}_\mu(P)] = \{x_1x_2\}$, whereas satisfaction of IC_4 requires the result to be $\{x_1, x_1x_2\}$. The same result is obtained for $\Delta^{\mathrm{AV}}_\mu(P)$, but $[\Delta^{\mathrm{bPAV}}_\mu(P)] = \{x_1, x_1x_2\}$.*

To curb the inflationary tendencies of Δ^{PAV}, operator Δ^{bPAV} introduces a penalty on interpretations depending on their size, in the process ensuring satisfaction of postulate IC_4 as well. Indeed, as Section 5 shows, Δ^{bPAV} is the *only* operator from a fairly broad class that manages to balance proportionality and fairness, as formalized by postulate IC_4.

A related problem with Δ^{PAV} stems from the fact that $s_{\mathrm{PAV}}(v, w)$ is obtained by counting only atoms v and w have in common. Hence, Δ^{PAV} is insensitive to the presence of extraneous, possibly unwanted atoms in w, the assumption being that atoms in w that are not in v represent issues on which v has no opinion on, and thus their presence has no effect on the satisfaction of v (see Example 4). This assumption has the side-effect of interfering with postulate IC_2.

Example 4. *It holds that $s_{\mathrm{PAV}}(x_1, x_1) = s_{\mathrm{PAV}}(x_1, x_1x_2)$, i.e., the presence of x_2 does not affect the satisfaction of x_1, which leads to non-satisfaction of IC_2. For $\mathcal{A} = \{x_1, x_2\}$, $P = (\varphi)$, where $[\varphi] = \{x_1\}$ and $\mu = \top$, we obtain that $[\Delta^{\mathrm{PAV}}_\mu(P)] = \{x_1, x_1x_2\}$, whereas satisfaction of postulate IC_2 would require the result to be $\{x_1\}$.*

Note that $s_{\mathrm{hD}}(x_1, x_1) = s_{\mathrm{hH}}(x_1, x_1) = h(2)$, while $s_{\mathrm{hD}}(x_1, x_1x_2) = s_{\mathrm{hH}}(x_1, x_1x_2) = h(1)$. Thus, according to s_{hD} and s_{hH}, x_1x_2 provides less satisfaction to x_1 than

x_1 alone. Consequently, for P and μ as above, $[\Delta^{\mathrm{hD}}_{\mu}(P)] = [\Delta^{\mathrm{hH}}_{\mu}(P)] = \{x_1\}$. This is in accord with postulate IC_2.

The *binary satisfaction-based* operators, consisting of the *harmonic drastic operator* Δ^{D} and the *harmonic Hamming operator* Δ^{H}, are introduced in an attempt to deal with the effect of unwanted atoms while, at the same time, providing proportional outcomes. The satisfaction measures they are based on penalize interpretations w for including additional, purportedly unwanted atoms. This is done by inverting familiar notions of distance, which pay attention to atoms appearing in one of the interpretation but not in the other. The harmonic function h is added to the satisfaction notion thus obtained with the idea of ensuring proportionality. The operators that emerge are worth investigating: neither of them extends PAV (as hinted at in Example 4), but from this point onward their properties diverge. Though Δ^{hH} does not extend PAV, it still ends up having interesting proportionality properties, formalized in Section 6. On the other hand, operator Δ^{hD} turns out to be so coarse in its assessment of satisfaction as to become, as Proposition 1 shows, indistinguishable from existing merging operators defined using drastic distance d_{D}. As a result, the Δ^{hD} operator is not responsive to proportionality requirements.

Proposition 1. *The satisfaction-based operator Δ^{hD} is equivalent to the distance-based operator Δ^{D}.*

What emerges is a landscape with three merging operators relevant to the issue of proportionality, i.e., Δ^{PAV}, Δ^{bPAV} and Δ^{hH}. Out of these, Δ^{bPAV} and Δ^{hH} address, each in its own way, problems arising with the Δ^{PAV} operator: Δ^{bPAV} deals with interpretations of varying sizes, while Δ^{hH} incorporates sensitivity toward rejected atoms. As we will see in Sections 5 and 6, the proposed solutions involve various trade-offs between proportionality and the IC postulates.

5 IC Postulates: Possibility and Impossibility

In this section we look at the merging operators introduced in Section 4 in light of the standard merging postulates IC_{0-8}. The first result shows that any satisfaction-based operator satisfies a core set of IC-postulates.

Proposition 2. *If s is a satisfaction measure, then the merging operator Δ^s satisfies postulates $\mathsf{IC}_{0-1,3,5-8}$.*

Proposition 2 applies to both the approval-based and the harmonic distance-based operators. What remains, then, is an understanding of how the new satisfaction measures interact with postulates IC_2 and IC_4, and we settle the issue by characterizing the types of satisfaction measures compliant with these postulates. If $v \neq w$, the following properties prove to be relevant:

$$(\mathsf{S}_1)\ s(v,v) > s(v,w); \quad (\mathsf{S}_2)\ s(v,v) > s(w,v);$$
$$(\mathsf{S}_3)\ s(v,v) = s(w,w); \quad (\mathsf{S}_4)\ s(v,w) = s(w,v).$$

They formalize the intuition that satisfaction is symmetric (S_4), maximal when one gets *exactly* what one wants, and trailing off as the outcome diverges from one's most desired outcome (S_{1-3}). Theorem 1 shows that properties S_{1-3} capture satisfaction measures compliant with postulate IC_2.

Theorem 1. *A satisfaction-based merging operator Δ^s satisfies postulate IC_2 iff s satisfies properties S_{1-3}.*

Since the satisfaction measures s_{AV}, s_{PAV} or s_{bPAV} satisfy none of the properties S_{1-3}, Theorem 1 implies that the approval-based operators Δ^{AV}, Δ^{PAV} or Δ^{bPAV} do not satisfy postulate IC_2. On the other hand, the satisfaction measures s_{hD} and s_{hH} do satisfy properties S_{1-3}, showing that the corresponding operators satisfy postulate IC_2.

As mentioned in Section 4, we do not require satisfaction measures to be symmetric and, indeed, s_{bPAV} is not symmetric (though the other satisfaction measures are). The following result shows that, in the presence of postulate IC_2, symmetry is connected to postulate IC_4.

Theorem 2. *If a satisfaction-based merging operator Δ^s satisfies postulate IC_2, then Δ^s satisfies postulate IC_4 if and only if s also satisfies property S_4 (i.e., is symmetric).*

Since the satisfaction measures s_{hD} and s_{hH} are symmetric and, as implied by Theorem 1, satisfy properties S_{1-3}, we get by Theorem 2 that they also satisfy postulate IC_4. Together with Proposition 2, this yields the full picture for the binary satisfaction-based operators Δ^{hH} and Δ^{hD}

Corollary 1. *The operators Δ^{hH} and Δ^{hD} satisfy postulates IC_{0-8}.*

For the approval-based operators, satisfaction of postulates IC_2 and IC_4 is clarified by another perspective on satisfaction measures. A satisfaction measure s is a *counting index* if there exists a function $\sigma\colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}$, called *the witness of* s, such that $\sigma(0,0) = 0$ and $s(v,w) = \sigma(|v \cap w|, |w|)$, for any interpretations v and w. Theorem 3 shows that counting indices do not fit with postulate IC_2.

Theorem 3. *If s is a counting index, the satisfaction-based merging operator Δ^s does not satisfy postulate IC_2.*

It is straightforward to see that the approval-based satisfaction measures introduced in Section 4 are counting indices. Thus, by Theorem 3, none of the operators they generate satisfies postulate IC_2. For postulate IC_4, however, the situation is different. Example 3 shows that the Δ^{AV} and Δ^{PAV} operators do not satisfy postulate IC_4, though Δ^{bPAV} manages to evade the counter-example. In fact, it turns out that not only does the operator Δ^{bPAV} satisfy postulate IC_4, but a much stronger result can be shown: it is the *only* operator based on a counting index that does so.

Theorem 4. *If Δ^s is a satisfaction-based merging operator such that s is a counting index with σ as witness, extends* PAV *and satisfies postulate IC_4, then $\sigma(x,y) = 2h(x) - h(y)$, for any $x, y \in \mathbb{R}$.*

It deserves emphasis that Δ^{bPAV} manages to satisfy postulate IC_4 even though s_{bPAV} is not a symmetric satisfaction measure: since Δ^{bPAV} does not satisfy postulate IC_2, Theorem 2 does not apply. Indeed, none of the approval-based operators manages to satisfy both postulates IC_2 and IC_4. This suggests that there is a trade-off between the kind of proportionality these operators stand for and these postulates.

One aspect that proves to be relevant is the fact that approval-based operators can consider interpretations of various sizes. Reflection on Examples 4 and 3 shows that the problematic situations always involve interpretations of different sizes. Interestingly, it turns out that fixing the size of the models of the constraint μ yields merging operators that behave well w.r.t. the IC postulates.

Theorem 5. *If all models of the constraint μ have some fixed size k, then the approval-based merging operators Δ^{AV}, Δ^{PAV} and Δ^{bPAV} satisfy all postulates IC_{0-8}.*

6 Two Types of Proportionality

Here we formalize two notions of proportionality, arising out of two different ways of conceptualizing satisfaction with respect to a possible outcome. To simplify the presentation, we define these notions only for very restricted profiles.

A formula φ is *complete* if it has exactly one model, and a profile P is complete if all the formulas in it are complete. We write $P = (v_1, \dots, v_n)$ to denote the complete profile with $[\varphi_i] = \{v_i\}$, for all $i \in \{1, \dots, n\}$. A complete profile $P = (v_1, \dots, v_n)$ is *simple* if $v_1 \cup \cdots \cup v_n = \mathcal{A}$, and either $v_i = v_j$ or $v_i \cap v_j = \emptyset$, for every $i, j \in \{1, \dots, n\}$.[1] A complete profile $P = (v_1, \dots, v_n)$ is *ℓ-simple* if it is simple and $|\{v_1, \dots, v_n\}| = \ell$, i.e., P contains ℓ distinct sets. If $v_1, \dots, v_\ell$ constitutes a partition of $\mathcal{A}$, and $p_1, \dots, p_\ell$ are positive integers, we write $(v_1^{p_1}, \dots, v_\ell^{p_\ell})$ to denote the ℓ-simple profile: $(\underbrace{v_1, \dots, v_1}_{p_1 \text{ times}}, \underbrace{v_2, \dots, v_2}_{p_2 \text{ times}}, \dots, \underbrace{v_\ell, \dots, v_\ell}_{p_\ell \text{ times}})$. If $P = (v_1^{p_1}, \dots, v_\ell^{p_\ell})$ is an ℓ-simple profile with $\sum_{i=1}^{\ell} p_i = n$, we say that P is *k-integral* if $\frac{k \cdot p_i}{n}$ is an integer, for every $i \in \{1, \dots, \ell\}$. Intuitively, for a model w of μ of size k, the fraction $\frac{k \cdot p_i}{n}$ denotes the intended satisfaction if proportionality is taken into account: out of the k atoms selected, the share of group i should be the relative size of the group.

We propose two proportionality postulates, formulated for simple profiles $P = (v_1^{p_1}, \dots, v_\ell^{p_\ell})$. As before, constraint μ_k has as its models all interpretations of size k.

($\mathsf{IC}_{\mathsf{cp}}$) For any $k \in \{1, \dots, m\}$ and $w \in [\Delta_{\mu_k}(P)]$, it holds that if P is k-integral and $|v_j| \geq \frac{k \cdot p_j}{n}$ for each j, $1 \leq j \leq l$, then $|v_i \cap w| = \frac{k \cdot p_i}{n}$, for all $i \in \{1, \dots, \ell\}$.

[1] In the context of ABC voting, such profiles are referred to as party-list profiles [21].

($\mathsf{IC_{bp}}$) If $P = (v_1^{p_1}, v_2^{p_2})$ is simple and there is a $w \in [\mu]$ s.t.

$$m - d_{\mathrm{H}}(v_i, w) = \frac{m \cdot p_i}{n} \text{ for } i \in \{1, 2\}, \tag{1}$$

then (1) holds for all $w' \in [\Delta_\mu(P)]$.

We refer to $\mathsf{IC_{cp}}$ and $\mathsf{IC_{bp}}$ as postulates of *weak classical proportionality* and *weak binary proportionality*, respectively, as they refer to different sources of satisfaction. Postulate $\mathsf{IC_{cp}}$ talks about *classical satisfaction*, in which agent i's satisfaction with an interpretation w is given by $|v_i \cap w|$, just like the satisfaction with a committee in an ABC election is measured by the number of approved committee members. This is the kind of satisfaction notion typically used in a social choice context. Postulate $\mathsf{IC_{bp}}$ talks about *binary satisfaction*, in which agent i's satisfaction with w is given by $m - d_{\mathrm{H}}(v_i, w)$, i.e., by the degree of closeness between v_i and w. This type of satisfaction, alluded to already in Section 4, follows from a logical viewpoint where positive and negative variable assignments are treated equally. This approach is better suited to deal with interpretations of varying sizes than the classical one, and thus postulate $\mathsf{IC_{bp}}$ allows such interpretations to be selected.

Intuitively, both postulates stipulate 'shares' groups of agents shall receive (under a classical or binary viewpoint) that meet proportionality based on the relative size of the groups. For $\mathsf{IC_{cp}}$ we restrict to μ_k, with k atoms to be distributed proportionally by each solution w (like for ABC elections). Postulate $\mathsf{IC_{bp}}$ states that in the presence of at least one admissible $w \in [\mu]$ that meets the proportionality requirements, *all* solutions shall meet said requirements (otherwise μ permits no proportional solution). Note that if $P = (v_1^{p_1}, v_2^{p_2})$ satisfies the conditions of $\mathsf{IC_{bp}}$, then P is m-integral, and the binary satisfaction of v_1 and v_2 adds up to m, i.e., $m - d_{\mathrm{H}}(v_1, w) + m - d_{\mathrm{H}}(v_2, w) = m$. Postulate $\mathsf{IC_{bp}}$ demands that this total satisfaction m is split proportionally.

Example 5. *For $\mathcal{A} = X \cup Y$, with $X = \{x_1, \ldots, x_6\}$ and $Y = \{y_1, y_2\}$, take the simple profile $P = (v_1^3, v_2^1)$, with $v_1 = x_1 \ldots x_6$ and $v_2 = y_1 y_2$, and a constraint μ_4, with models of size 4. It is straightforward that, according to $\mathsf{IC_{cp}}$, an optimal outcome contains three variables from X and one from Y, e.g., the interpretation $w = x_1x_2x_3y_1$. Such an outcome is in the spirit of classical proportionality.*

According to postulate $\mathsf{IC_{bp}}$, an optimal outcome w would be such that $d_{\mathrm{H}}(v_1, w) = 2$ and $d_{\mathrm{H}}(v_2, w) = 6$, e.g., $w' = x_1x_2x_3x_4$, $w'' = x_1x_2x_3x_4x_5y_1$ or $w''' = x_1x_2x_3x_4x_5x_6y_1y_2$. Note that $\mathsf{IC_{bp}}$ allows interpretations of varying sizes to be selected. If the size is restricted to 4 (i.e., the constraint is μ_4), then the outcome narrows down to interpretations such as w', which consist of four atoms from X. More details on this can be found in the supplement.

Example 5 shows that classical and binary proportionality may require different interpretations to be selected on the same input. Thus, even though our notions of proportionality apply only to simple profiles, they set up a clear boundary for distinguishing among the different merging operators.

Theorem 6. *The merging operators Δ^{PAV} and Δ^{bPAV} satisfy postulate $\mathsf{IC}_{\mathsf{cp}}$, Δ^{hH} satisfies postulate $\mathsf{IC}_{\mathsf{bp}}$, while $\Delta^{\mathrm{H},\Sigma}$, $\Delta^{\mathrm{H,gmax}}$, Δ^{hD} and Δ^{AV} satisfy neither $\mathsf{IC}_{\mathsf{cp}}$ nor $\mathsf{IC}_{\mathsf{bp}}$.*

The proposed merging operators Δ^{PAV} and Δ^{bPAV} are representative of the notion of classical proportionality, while Δ^{hH} is representative for binary proportionality. Theorem 7 shows that these notions are thoroughly incompatible.

Theorem 7. *There is no merging operator that satisfies IC_1 and both $\mathsf{IC}_{\mathsf{cp}}$ and $\mathsf{IC}_{\mathsf{bp}}$.*

7 Computational Complexity

To investigate the complexity of our novel merging operators, we look at the standard decision problem studied in this context [20]: given an operator Δ, a profile P, an integrity constraint μ, and a Boolean formula ψ, determine whether $\Delta_\mu(P) \models \psi$ holds. That is, the decision problem asks whether a formula ψ follows from the merged result. The hardness results we use and recall here hold when $\psi = a$ is an atom. The two main complexity classes appearing here are Δ_2^P and Θ_2^P, denoting the classes of decision problems solvable via a deterministic polynomial time algorithm with access to an NP oracle, with the latter class having the additional restriction that at most logarithmically many oracle calls may be made. Many standard merging operators are complete for one of these two classes [20].

We show that our novel operators fit into this picture; we obtain Θ_2^P hardness and Δ_2^P membership for all new operators, except for Δ^{AV}, which we show to be Θ_2^P-complete. That is, our introduction of proportionality leads to neither milder nor significantly more complex operators. Hardness for Θ_2^P can be shown by adapting an existing reduction, originally from belief revision [7, Theorem 6.9]. Finally, membership diverges for Δ^{hH}, Δ^{PAV}, Δ^{bPAV} and Δ^{AV}, since the first three operators induce an exponential set of possible satisfaction scores for interpretations—in contrast to Δ^{AV} that only induces a polynomial set.

Theorem 8. *Deciding whether a formula follows from the result of merging operator Δ^s is Θ_2^P-complete for $s = \mathrm{AV}$, and both Θ_2^P-hard and in Δ_2^P, for $s \in \{\mathrm{PAV}, \mathrm{bPAV}, \mathrm{hH}\}$.*

We conjecture that merging under Δ^{hH}, Δ^{PAV} and Δ^{bPAV} is Δ_2^P-complete. However, usual hardness reductions used to show Δ_2^P-hardness in merging are not suited for our novel operators based on harmonic functions.

8 Applications Beyond Belief Merging

In this section we briefly discuss how our results can be transferred to other, related formalisms.

Variable Approval-Based Committee Elections

In contrast to ABC elections as introduced in Section 3, it is sometimes desirable to have flexibility with respect to the size of the committee by not fixing its size in advance [16, 14]. We refer to ABC voting rules without a size constraint as *variable ABC voting rules*. Note that, as mentioned in Section 4, a merging operator defines a variable ABC rule by setting $\mu = \top$. It is easy to see that the AV and PAV operators are not sensible in this context, as $w = \mathcal{A}$ (i.e., setting all atoms to true) is always an optimal model. However, the Δ^{bPAV} and Δ^{hH} operators present themselves as novel additions to this framework, being *proportional* variable ABC rules.

Goal-Based Voting

Goal-based voting [26] is a formalism similar to belief merging but with a focus on resolute rules (i.e., rules return only one model) and with different postulates. All proposed operators in this paper can be viewed as goal-based voting rules (subject to tie-breaking), and our proportionality postulates can be adapted for this setting as well. To the best of our knowledge, our proposed merging operators yield the first proportional goal-based voting rules. It would be particularly interesting to see whether Theorem 4 can be replicated by axioms from the goal-based voting setting (instead of postulate IC_4).

Judgment Aggregation

Judgment aggregation (JA) is another formalism for aggregating beliefs, related but distinct from belief merging, with the connection having been discussed [11]. Even though they differ in important aspects, the main ideas in our paper can be transferred from belief merging to JA. While propositional variables are the basic building blocks for belief merging, it might be more suitable to take the agenda (a set of propositional formulas) as the basis for defining proportionality in JA. This allows for the definition of proportional JA operators. Further work is required to analyze the resulting JA operators.

9 Discussion

In this paper we have initiated the study of proportional belief merging operators. We have presented three proportional operators: the PAV operator and the bounded PAV operator, both satisfying $\mathsf{IC}_{\mathsf{cp}}$, and the harmonic Hamming operator satisfying $\mathsf{IC}_{\mathsf{bp}}$. We summarize our results in Table 3.

Apart from the questions posed in Section 8, the current work suggests several directions for future research, While the two proportionality postulates we proposed apply only to certain instances, even weak proportionality postulates have proven sufficient for axiomatic characterizations [21] and in our paper these two postulates are sufficient to distinguish proportional from non-proportional operators. On the other hand, stronger postulates are desirable to determine to

Table 3: Summary of results. New results in gray, for all others see Konieczny et al. (2004). Per Theorem 5, for results marked with $*$ the $\times$ becomes ✓ when models of the constraint μ are assumed to have fixed size.

	$\mathsf{IC}_{0,1,3,5-8}$	IC_2	IC_4	$\mathsf{IC}_{\mathsf{cp}}$	$\mathsf{IC}_{\mathsf{bp}}$	Complexity
$\Delta^{\mathrm{H},\Sigma}$	✓	✓	✓	$\times$	$\times$	Θ_2^P-c
$\Delta^{\mathrm{H,gmax}}$	✓	✓	✓	$\times$	$\times$	Δ_2^P-c
$\Delta^{\mathrm{hD}} \equiv \Delta^{\mathrm{D}}$	✓	✓	✓	$\times$	$\times$	Θ_2^P-c
Δ^{hH}	✓	✓	✓	$\times$	✓	in Δ_2^P, Θ_2^P-h
Δ^{AV}	✓	$\times^*$	$\times^*$	$\times$	$\times$	Θ_2^P-c
Δ^{PAV}	✓	$\times^*$	$\times^*$	✓	$\times$	in Δ_2^P, Θ_2^P-h
Δ^{bPAV}	✓	$\times^*$	✓	✓	$\times$	in Δ_2^P, Θ_2^P-h

which degree proportionality guarantees can be given. This has recently been investigated in the context of approval-based committee elections [1, 2, 28]; this line of work can serve as a basis for a similar analysis for belief merging operators.

Finally, manipulation and strategic voting, common concerns in social choice theory, have received some attention in the belief merging framework as well [10, 15]. It can be expected that proportional belief merging operators are prone to strategic voting, as in ABC voting even weak forms of proportionality and strategy-proofness have been shown to be incompatible [27]. Still, it has been found that the percentage of manipulable instances depends strongly on the choice of voting rules [22], indicating that a detailed analysis of vulnerabilities is an interesting avenue for future work.

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Appendix

In this appendix we give further examples that clarify our proposed operators and their properties (Section A), as well as give proof detail for the formal statements in the paper (Section B), and provide an additional discussion on the behavior of the bounded PAV merging operator (Section C).

A Examples

Example 6 involves all the operators we talk about in the paper (both the standard distance-based ones and the satisfaction-based operators we introduce), together with tables for the computed distances and satisfactions. It is a companion to Examples 1 and 2 in the main body of the paper, and it expands on the main ideas and techniques.

Example 6. *For the set of atoms $\mathcal{A} = X \cup Y$, with $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5\}$, take the simple profile $P = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$, with $[\varphi_1] = [\varphi_2] = [\varphi_3] = \{x_1x_2x_3x_4x_5\}$ and $[\varphi_4] = \{y_1y_2y_3y_4y_5\}$, and a constraint μ_5, with models of size 5. The distances between the formulas in P and the models of μ (i.e., $d_{\mathrm{H}}(\varphi_i, w)$ and $d_{\mathrm{D}}(\varphi_i, w)$, for $i \in \{1, 2, 3, 4\}$ and $w \in [\mu]$), the satisfactions of the formulas in P with the models of μ (i.e., $s_{\mathrm{AV}}(\varphi_i, w)$, $s_{\mathrm{PAV}}(\varphi_i, w)$, $s_{\mathrm{bPAV}}(\varphi_i, w)$ and $s_{\mathrm{AV}}(\varphi_i, w)$, for $i \in \{1, 2, 3, 4\}$ and $w \in [\mu]$), as well as the aggregated distances and satisfactions, are depicted in Tables 4, 5 and 6. In these tables we only focus on a representative sample of models of μ. Optimal values are written in bold font.*

For the distance-based operators $\Delta^{\mathrm{H},\Sigma}$, $\Delta^{\mathrm{H,gmax}}$, $\Delta^{\mathrm{D},\Sigma}$ and $\Delta^{\mathrm{D,gmax}}$ note that optimal values are those that minimize overall distance to P. Thus, we have that $d^{\Sigma}_{\mathrm{H}}(P, x_1x_2x_3x_4x_5) < d^{\Sigma}_{\mathrm{H}}(P, x_1x_2x_3x_4y_1)$, which means that $x_1x_2x_3x_4x_5$ is, according to the aggregation function Σ, closer to P than $x_1x_2x_3x_4y_1$. On the other hand, it holds that $d^{\mathrm{gmax}}_{\mathrm{H}}(P, x_1x_2x_3y_1y_2) < d^{\mathrm{gmax}}_{\mathrm{H}}(P, x_1x_2x_3x_4y_1)$, since $(6, 4, 4, 4, 4)$ lexicographically dominates $(8, 2, 2, 2, 2)$. We obtain that:

$$\begin{aligned}[\Delta^{\mathrm{H},\Sigma}_{\mu}(P)] &= [\Delta^{\mathrm{D},\Sigma}_{\mu}(P)] \\ &= [\Delta^{\mathrm{D,gmax}}_{\mu}(P)] \\ &= \{x_1x_2x_3x_4x_5\},\end{aligned}$$

while the optimal outcomes according to $\Delta^{\mathrm{H,gmax}}$ are interpretations containing three atoms from X and two from Y, such as $x_1x_2x_3y_1y_2$. It is straightforward to see, looking at Table 4, that $\Delta^{\mathrm{D},\Sigma}$ and $\Delta^{\mathrm{D.gmin}}$ are equivalent, since they both do the same thing, i.e., count the number of bases of which w is not a model.

For the approval-based operators Δ^{AV}, Δ^{PAV} and Δ^{bPAV} and the binary satisfaction-based operator Δ^{hH}, keep in mind that optimal values are those that maximize overall satisfaction with respect to P. We obtain that:

$$[\Delta^{\mathrm{AV}}_{\mu}(P)] = \{x_1x_2x_3x_4x_5\},$$

	d_{H}				d_{D}			
	$4 \cdot x_1x_2x_3x_4x_5$	$y_1y_2y_3y_4y_5$	Σ	gmax	$4 \cdot x_1x_2x_3x_4x_5$	$y_1y_2y_3y_4y_5$	Σ	gmax
$x_1x_2x_3x_4x_5$	$4 \cdot 0$	10	**10**	$(10, 0, 0, 0, 0)$	$4 \cdot 0$	1	**1**	$(\mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$
$x_1x_2x_3x_4y_1$	$4 \cdot 2$	8	16	$(8, 2, 2, 2, 2)$	$4 \cdot 1$	1	5	$(1, 1, 1, 1, 1)$
$x_1x_2x_3y_1y_2$	$4 \cdot 4$	6	22	$(\mathbf{6}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4})$	$4 \cdot 1$	1	5	$(1, 1, 1, 1, 1)$
$x_1x_2y_1y_2y_3$	$4 \cdot 6$	4	28	$(6, 6, 6, 6, 4)$	$4 \cdot 1$	1	5	$(1, 1, 1, 1, 1)$
$x_1y_1y_2y_3y_4$	$4 \cdot 8$	2	34	$(8, 8, 8, 8, 2)$	$4 \cdot 1$	1	5	$(1, 1, 1, 1, 1)$
$y_1y_2y_3y_4y_5$	$4 \cdot 10$	0	40	$(10, 10, 10, 10, 0)$	$4 \cdot 1$	0	4	$(1, 1, 1, 1, 0)$
$\ldots$								

Table 4: Distances d_{H} and d_{D}, as well as the overall distances aggregated with Σ and gmin, for profile P and constraint μ from Example 6.

	s_{AV}			s_{PAV}		
	$4 \cdot x_1x_2x_3x_4x_5$	$y_1y_2y_3y_4y_5$	Σ	$4 \cdot x_1x_2x_3x_4x_5$	$y_1y_2y_3y_4y_5$	Σ
$x_1x_2x_3x_4x_5$	$4 \cdot 5$	0	**20**	$4 \cdot h(5)$	$h(0)$	9.13
$x_1x_2x_3x_4y_1$	$4 \cdot 4$	1	17	$4 \cdot h(4)$	$h(1)$	**9.33**
$x_1x_2x_3y_1y_2$	$4 \cdot 3$	2	14	$4 \cdot h(3)$	$h(2)$	8.83
$x_1x_2y_1y_2y_3$	$4 \cdot 2$	3	11	$4 \cdot h(2)$	$h(3)$	7.83
$x_1y_1y_2y_3y_4$	$4 \cdot 1$	4	8	$4 \cdot h(1)$	$h(4)$	6.08
$y_1y_2y_3y_4y_5$	$4 \cdot 0$	5	5	$4 \cdot h(0)$	$h(5)$	2.28
$\ldots$						

Table 5: Satisfactions s_{AV} and s_{PAV}, as well as the overall satisfactions, for profile P and constraint μ from Example 6.

while the optimal outcomes according to the operators Δ^{PAV}, Δ^{bPAV} and Δ^{hH} are interpretations containing four atoms from X and one atom from Y, such as $x_1x_2x_3x_4y_1$. The distribution of chosen atoms according to the latter operators reflects the ratio of supporters of X to supporters of Y in P, which is $4:1$.

It is immediately visible how the operator $\Delta^{\mathrm{H},\Sigma}$ favors the majority option, while the operator $\Delta^{\mathrm{H,gmax}}$ optts for the more egalitarian outcome. Neither of these strategies is guaranteed to yield outcomes that reflect the proportion of support for a particular issue in the profile. By contrast, the operators Δ^{PAV}, Δ^{bPAV} and Δ^{hH} aim for proportional outcomes, and this is visible on the example.

	s_{bPAV}			s_{hH}		
	$4 \cdot x_1x_2x_3x_4x_5$	$y_1y_2y_3y_4y_5$	Σ	$4 \cdot x_1x_2x_3x_4x_5$	$y_1y_2y_3y_4y_5$	Σ
$x_1x_2x_3x_4x_5$	$4 \cdot (2 \cdot h(5) - h(5))$	$2 \cdot h(0) - h(5)$	6.85	$4 \cdot h(10-0)$	$h(10-10)$	11.71
$x_1x_2x_3x_4y_1$	$4 \cdot (2 \cdot h(4) - h(5))$	$2 \cdot h(1) - h(5)$	**7.25**	$4 \cdot h(10-2)$	$h(10-8)$	**12.37**
$x_1x_2x_3y_1y_2$	$4 \cdot (2 \cdot h(3) - h(5))$	$2 \cdot h(2) - h(5)$	6.25	$4 \cdot h(10-4)$	$h(10-6)$	11.88
$x_1x_2y_1y_2y_3$	$4 \cdot (2 \cdot h(3) - h(5))$	$2 \cdot h(3) - h(5)$	4.25	$4 \cdot h(10-6)$	$h(10-4)$	10.78
$x_1y_1y_2y_3y_4$	$4 \cdot (2 \cdot h(1) - h(5))$	$2 \cdot h(4) - h(5)$	0.75	$4 \cdot h(10-8)$	$h(10-2)$	8.72
$y_1y_2y_3y_4y_5$	$4 \cdot (2 \cdot h(0) - h(5))$	$2 \cdot h(5) - h(5)$	-6.85	$4 \cdot h(10-10)$	$h(10-0)$	2.93
$\ldots$						

Table 6: Satisfactions s_{bPAV} and s_{hH}, as well as the overall satisfactions, for profile P and constraint μ from Example 6.

	s_{PAV}			s_{hH}		
	$3 \cdot x_1x_2x_3x_4x_5x_6$	y_1y_2	Σ	$3 \cdot x_1x_2x_3x_4x_5x_6$	y_1y_2	Σ
$x_1x_2x_3x_4$	$3 \cdot h(4)$	$h(0)$	6.25	$3 \cdot h(6)$	$h(2)$	**8.85**
$x_1x_2x_3y_1$	$3 \cdot h(3)$	$h(1)$	**6.5**	$3 \cdot h(4)$	$h(4)$	8.33
$x_1x_2y_1y_2$	$3 \cdot h(2)$	$h(2)$	6.0	$3 \cdot h(2)$	$h(2)$	6.95
...						

Table 7: Satisfaction s_{PAV} and s_{hH}, as well as the aggregates satisfactions, for profile P and constraint μ in Examples 5 and 7.

Example 7 presents a more elaborate version of Example 5, used to highlight the difference between classical and binary proportionality.

Example 7 (Expanded from Example 5)**.** *For the set of atoms $\mathcal{A} = X \cup Y$, with $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $Y = \{y_1, y_2\}$, take the simple profile $P = (v_1^3, v_2^1)$, with $v_1 = x_1x_2x_3x_4x_5x_6$ and $v_4 = y_1y_2$, and a constraint μ_4, with models of size 4. Since the number of formulas in P is 4 (i.e., $n = 4$), it is easy to see that the profile P is 4-integral as well as 8-integral (the latter is needed because $m = 8$ in this case).*

We would like to understand what kind of interpretations would be chosen by a merging operator Δ that satisfies postulate $\mathsf{IC_{cp}}$ and $\mathsf{IC_{bp}}$, respectively. To do this, we can use the equalities presented in these postulates to infer properties of an optimal outcome, i.e., an interpretation w such that $w \in [\Delta_\mu(P)]$. In these equalities, we are basically treating w as an unknown and solving for it.

Assume, first, we are working with a merging operator Δ that satisfies postulate $\mathsf{IC_{cp}}$. Thus, for $k = 4$, postulate $\mathsf{IC_{cp}}$ tells us that if $w \in [\Delta_{\mu_4}(P)]$, then it holds that:

$$\begin{aligned} |v_1 \cap w| &= \frac{4 \cdot 3}{4} \\ &= 3, \end{aligned}$$

and:

$$\begin{aligned} |v_2 \cap w| &= \frac{4 \cdot 1}{4} \\ &= 1. \end{aligned}$$

Thus, from the standpoint of classical proportionality, an optimal outcome of size 4 reflects the proportion of agents that approve atoms within it, and it would contain three variables from X and one from Y, e.g., the interpretation $w = x_1x_2x_3y_1$.

Assume, however, that we are working with a merging operator Δ that satisfies postulate $\mathsf{IC_{bp}}$. Postulate $\mathsf{IC_{bp}}$ tells us that if there exists an interpretation $w' \in [\Delta_{\mu_4}(P)]$ that satisfies the equality:

$$m - d_{\mathrm{H}}(v_i, w) = \frac{m \cdot p_i}{n},$$

for $i \in \{1,2\}$, then every interpretation in $[\Delta_\mu(P)]$ satisfies this equality. Suppose there exists such an interpretation w. The equality requires that:

$$\begin{aligned} 8 - d_{\mathrm{H}}(v_1, w) &= \frac{8 \cdot 3}{4} \\ &= 6, \end{aligned}$$

and:

$$\begin{aligned} 8 - d_{\mathrm{H}}(v_2, w) &= \frac{8 \cdot 1}{4} \\ &= 2. \end{aligned}$$

This implies that $d_{\mathrm{H}}(v_1, w) = 2$ and $d_{\mathrm{H}}(v_2, w) = 6$. Among all possible interpretations, the ones that satisfy these conditions are:

(a) interpretations of size 4, consisting of four atoms from X, e.g., $x_1x_2x_3x_4$;

(b) interpretations of size 6, consisting of five atoms from X and one from Y, e.g., $x_1x_2x_3x_4x_5y_1$;

(c) the interpretation consisting of all atoms from X and all atoms from Y, i.e., $x_1x_2x_3x_4x_5x_6y_1y_2$.

Thus, postulate $\mathsf{IC}_{\mathsf{bp}}$ says that if at least one of these interpretations are in mods μ, then the interpretations that make it into the result are all from the same list.

Note that from the standpoint of binary proportionality, it makes sense to select among interpretations of varying sizes, as the satisfaction notion is calibrated to take into account the differences that arise. Note also that if the constraint is restricted to interpretations of size 4 (i.e., the constraint is μ_4), then only interpretations of type (a) get selected. In this setup, an interpretation such as $w = x_1x_2x_3y_1$ provides less satisfaction to P than interpretations containing only atoms from X, such as $w' = x_1x_2x_3x_4$. This is because for agents v_1, v_2 and v_3 the exclusion of the desired atom x_4 at the expense of the undesired atom y_1 (when going from w' to w) incurs double the penalty as in the case of classical proportionality.

The quantity $|v_i \cap w|$ in $\mathsf{IC}_{\mathsf{cp}}$ is indicative of notions of classical satisfaction, while the quantity $m - d_{\mathrm{H}}(v_i, w)$ in $\mathsf{IC}_{\mathsf{bp}}$ is indicative of binary satisfaction. The operators Δ^{PAV} and Δ^{bPAV} are representatives of the former notion and the operator Δ^{hH} is representative of the latter. Notice that under the constraint μ_4 the operators Δ^{PAV} and Δ^{bPAV} select interpretations that have three atoms from X and one from Y, e.g., $x_1x_2x_3y_1$, while Δ^{hH} selects interpretations that have only atoms from X, e.g., $x_1x_2x_3x_4$. See Table 7 for an illustration.

B Proof Details

In this proof appendix we give proof details for the formal statements in the main paper, which were left out due to page limitations.

Proofs of Section 4

Proposition 3. *The approval-based merging operator* Δ^{AV} *does not extend* PAV.

Proof. For $\mathcal{A}$, P and μ as in Example 1, it holds that $[\Delta_\mu^{\mathrm{AV}}(P)] = \{x_1x_2x_3x_4\}$, whereas the PAV outcome in the corresponding ABC election, as shown in Example 2, outputs $x_1x_2x_3y_1$. □

Proposition 4. *The merging operators* Δ^{hD} *and* Δ^{hH} *do not extend* PAV.

Proof. For Δ^{hD} take $\mathcal{A}$, P and μ as in Example 1. It holds that $[\Delta_\mu^{\mathrm{hD}}(P)] = \{x_1x_2x_3x_4\}$, whereas the PAV outcome in the corresponding ABC election, as shown in Example 2, outputs $x_1x_2x_3y_1$.

For Δ^{hH} take $\mathcal{A} = X \cup Y$, where $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $Y = \{y_1, y_2\}$, the profile $P = (\varphi_1, \varphi_2)$, where $[\varphi_1] = \{x_1x_2x_3x_4x_5x_6\}$ and $[\varphi_2] = \{y_1y_2\}$, and the constraint μ such that $[\mu] = \{x_1x_2x_3x_4, x_1x_2x_3y_1, x_1x_2y_1y_2\}$. We obtain that $\Delta_\mu^{\mathrm{PAV}}(P) = \{x_1x_2x_3y_1\}$ (see Table 7 for a depiction of the satisfactions), and thus the corresponding PAV election for committees of size 4 would output $x_1x_2x_3y_1$. However, $\Delta_\mu^{\mathrm{hH}}(P) = \{x_1x_2x_3x_4\}$, which would not be among the committees elected by the corresponding ABC election. □

We arrive at Propostion 1 *via* some intermediary notions and results. First, we define the satisfaction measure s_{sD} as $s_{\mathrm{sD}}(v, w) = m - d_{\mathrm{D}}(v, w)$, for any interpretations v and w.

Lemma 1. *If* φ *is a formula and* w *is an interpretation, then it holds that:*

(a) if $w \in [\varphi]$*, then* $s_{\mathrm{sD}}(\varphi, w) = m$ *and* $s_{\mathrm{hD}}(\varphi, w) = h(m)$*;*

(b) if $w \notin [\varphi]$*, then* $s_{\mathrm{sD}}(\varphi, w) = m - 1$ *and* $s_{\mathrm{hD}}(\varphi, w) = h(m-1)$*.*

Proof. If $w \in [\varphi]$ and $v \in [\varphi]$, then by definition we have that $s_{\mathrm{sD}}(v, w) = m - d_{\mathrm{D}}(v, w)$, where m is the number of propositional atoms in $\mathcal{A}$. Thus, it holds that:

$$s_{\mathrm{sD}}(v, w) = \begin{cases} m, \text{ if } v = w, \\ m - 1, \text{ if } v \neq w. \end{cases}$$

Similarly, we obtain that:

$$s_{\mathrm{hD}}(v, w) = \begin{cases} h(m), \text{ if } v = w, \\ h(m-1), \text{ if } v \neq w. \end{cases}$$

If $w \in [\varphi]$, the maximal value of $s_{\mathrm{sD}}(v, w)$, for $v \in [\varphi]$, is m, which means that $s_{\mathrm{sD}}(\varphi, w) = m$. Similarly, the maximal value of $s_{\mathrm{hD}}(v, w)$, for $v \in [\varphi]$, is $h(m)$, which means that $s_{\mathrm{hD}}(\varphi, w) = h(m)$.

If $w \notin [\varphi]$, then $s_{\mathrm{sD}}(v, w) = m - 1$ and $s_{\mathrm{hD}}(v, w) = m - 1$, for every $v \in [\varphi]$. This means that $s_{\mathrm{sD}}(\varphi, w) = m - 1$ and $s_{\mathrm{hD}}(\varphi, w) = h(m-1)$. □

The satisfaction-based merging operator defined using the satisfaction measure s_{sD} is denoted as Δ^{sD}. We now show that Δ^{hD} and Δ^{sD} are equivalent.

Lemma 2. *The satisfaction-based operators Δ^{hD} and Δ^{sD} are equivalent.*

Proof. Take a profile $P = (\varphi_1, \ldots, \varphi_n)$ and two interpretations $w_1, w_2 \in [\mu]$. We will show that $s_{\mathrm{sD}}(P, w_1) \geq s_{\mathrm{sD}}(P, w_2)$ if and only if $s_{\mathrm{hD}}(P, w_1) \geq s_{\mathrm{hD}}(P, w_2)$.

We will denote by a_i the number of formulas φ in P such that $w_i \in [\varphi]$, and by b_i the number of formulas φ in P such that $w_i \notin [\varphi]$, for $i \in \{1, 2\}$. It then holds that $a_1 + b_1 = a_2 + b_2 = n$ and, by Lemma 1, that:

$$\begin{aligned}
s_{\mathrm{sD}}(P, w_i) &= a_i m + b_i(m-1), \\
s_{\mathrm{hD}}(P, w_i) &= a_i h(m) + b_i h(m-1),
\end{aligned}$$

for $i \in \{1, 2\}$. The claim we want to prove translates as:

$$\begin{aligned}
&a_1 m + b_1(m-1) \geq a_2 m + b_2(m-1) \text{ iff} \\
&a_1 h(m) + b_1 h(m-1) \geq a_2 h(m) + b_2 h(m-1).
\end{aligned}$$

With some algebraic manipulation of the left-hand-side term, and using the fact that $a_1 + b_1 = a_2 + b_2 = n$, we obtain that:

$$\begin{aligned}
&a_1 m + b_1(m-1) \geq a_2 m + b_2(m-1) \text{ iff} \\
&(a_1 + b_1)m - b_1 \geq (a_2 + b_2)m - b_2 \text{ iff} \\
&nm - b_1 \geq nm - b_2 \text{ iff} \\
&b_2 \geq b_1 \text{ iff} \\
&n - a_2 \geq n - a_1 \text{ iff} \\
&a_1 \geq a_2.
\end{aligned}$$

With some algebraic manipulation of the right-hand-side term, and using the facts that $h(m) = h(m-1) + \frac{1}{m}$ and $a_1 + b_1 = a_2 + b_2 = n$, we obtain that:

$$\begin{aligned}
&a_1 h(m) + b_1 h(m-1) \geq a_2 h(m) + b_2 h(m-1) && \text{iff} \\
&a_1(h(m-1) + \frac{1}{m}) + b_1 h(m-1) \geq && \\
&a_2(h(m-1) + \frac{1}{m}) + b_2 h(m-1) && \text{iff} \\
&(a_1 + b_1)h(m-1) + a_1(\frac{1}{m}) \geq && \\
&(a_2 + b_2)h(m-1) + a_2(\frac{1}{m}) && \text{iff} \\
&nh(m-1) + a_1(\frac{1}{m}) \geq nh(m-1) + a_2(\frac{1}{m}) && \text{iff} \\
&a_1 \geq a_2.
\end{aligned}$$

Thus, both sides reduce to the same inequality, and are therefore equivalent. Moreover, it is straightforward to see that equality is obtained on both sides in the same case: when there are as many formulas in P that feature w_1 as a model as there are formulas that feature w_2 as a model. In other words, we have that:

$$\begin{aligned} s_{\mathrm{sD}}(P, w_1) = s_{\mathrm{sD}}(P, w_2) &\text{ iff } s_{\mathrm{hD}}(P, w_1) = s_{\mathrm{hD}}(P, w_2) \\ &\text{ iff } a_1 = a_2. \end{aligned}$$

We have obtained, therefore, that $s_{\mathrm{sD}}(P, w_1) \geq s_{\mathrm{sD}}(P, w_2)$ if and only if $s_{\mathrm{hD}}(P, w_1) \geq s_{\mathrm{hD}}(P, w_2)$. This, now, implies the conclusion, namely that $\Delta^{\mathrm{hD}}_{\mu}(P) \equiv \Delta^{\mathrm{sD}}_{\mu}(P)$, for any constraint μ. □

Proposition 1. *The satisfaction-based operator Δ^{hD} is equivalent to the distance-based operator Δ^{D}.*

Proof. By Lemma 2, operator Δ^{hD} is equivalent to Δ^{sD} defined previously. It is now straightforward to see that Δ^{sD} is equivalent to Δ^{D}. □

Proofs of Section 5

Proposition 2. *If s is a satisfaction measure, then the merging operator Δ^s satisfies postulates $\mathsf{IC}_{0-1,3,5-8}$.*

Proof. Using the definition of the satisfaction-based operator Δ^s we infer that $\emptyset \subset [\Delta^s_{\mu}(P)] \subseteq [\mu]$, i.e., Δ^s is a formula whose set of models is a non-empty subset of the set of models of μ, which implies that postulates IC_{0-1} are satisfied. Since $\Delta^s_{\mu}(P)$ is defined solely in terms of its models, the syntax of the formulas involved does not influence the merging result and, hence, postulate IC_3 is satisfied.

For postulate IC_5, take an interpretation $w \in [\Delta^s_{\mu}(P_1) \wedge \Delta^s_{\mu}(P_2)]$, and an arbitrary interpretation $w' \in [\mu]$. We have that:

$$\sum_{\varphi \in P_1} s(\varphi, w) \geq \sum_{\varphi \in P_1} s(\varphi, w'), \tag{2}$$

$$\sum_{\varphi \in P_2} s(\varphi, w) \geq \sum_{\varphi \in P_2} s(\varphi, w'). \tag{3}$$

Adding the two inequalities gives us:

$$\sum_{\varphi \in P_1} s(\varphi, w) + \sum_{\varphi \in P_2} s(\varphi, w) \geq \sum_{\varphi \in P_1} s(\varphi, w') + \sum_{\varphi \in P_2} s(\varphi, w'),$$

which, in turn, implies that:

$$\sum_{\varphi \in (P_1 + P_2)} s(\varphi, w) \geq \sum_{\varphi \in (P_1 + P_2)} s(\varphi, w'). \tag{4}$$

Thus, the interpretation w, which provides maximal satisfaction for profiles P_1 and P_2, also provides maximal satisfaction for profile $P_1 + P_2$, which allows us to conclude that $w \in [\Delta^s_{\mu}(P_1 + P_2)]$.

	v	w	max
v	$s(v,v)$	$s(w,v)$	$\max\{s(v,v),s(w,v)\}$
w	$s(v,w)$	$s(w,w)$	$\max\{s(v,w),s(w,w)\}$

Figure 2: Satisfaction indices when $P=(\varphi)$, $[\varphi]=[\mu]=\{v,w\}$. The models of φ are written on the top row; the columns indicate models of μ.

For postulate IC_6 notice that if one of the inequalities 2 or 3 is strict, then inequality 4 is also strict. Thus, if interpretation w' does not provide maximal satisfaction with respect to P_1 or P_2, then it does not provide maximal satisfaction with respect to P_1+P_2 either. In other words, if $w' \notin [\Delta^s_\mu(P_1)\wedge\Delta^s_\mu(P_2)]$, then $w' \notin [\Delta^s_\mu(P_1+P_2)]$, which proves the claim.

For postulate IC_7, we have that if $w \in [\Delta^s_{\mu_1}(P)\wedge\mu_2]$, then $s(P,w)\geq s(P,w')$, for any $w'\in[\mu_1]$. Since $[\mu_1\wedge\mu_2]\subseteq[\mu_1]$, it is straightforward to conclude from here that $s(P,w)\geq s(P,w')$, for any $w'\in[\mu_1\wedge\mu_2]$, i.e., if w provides maximal satisfaction when the available options are the models of μ_1, it will also provide maximal satisfaction when we restrict the available options to the models of $\mu_1\wedge\mu_2$. Since $w\in[\mu_2]$ as well, it follows that $w\in[\Delta^s_{\mu_1\wedge\mu_2}(P)]$.

Conversely, for postulate IC_8, suppose $w\in[\Delta^s_{\mu_1\wedge\mu_2}(P)]$ and suppose $w\notin[\Delta^s_{\mu_1}(P)\wedge\mu_2]$. This means that $w\notin[\Delta^s_{\mu_1}(P)]$. Since $\Delta^s_{\mu_1}(P)\wedge\mu_2$ is consistent, there exists $w'\in[\Delta^s_{\mu_1}(P)\wedge\mu_2]$, which, together with the finding that $w\notin[\Delta^s_{\mu_1}(P)]$, implies that $s(P,w')>s(P,w)$. However, from the assumption that $w\in[\Delta^s_{\mu_1\wedge\mu_2}(P)]$ we obtain that $s(P,w)\geq s(P,w')$, which leads to a contradiction. □

Theorem 1. *A satisfaction-based merging operator Δ^s satisfies postulate IC_2 iff s satisfies properties S_{1-3}.*

Proof. ("⇒") Take a satisfaction-based merging operator Δ^s that satisfies postulate IC_2. We will show that s satisfies property S_{1-3}.

For property S_1, take interpretations v and w such that $v\neq w$. Consider, now, formulas φ and μ such that $[\varphi]=\{v\}$ and $[\mu]=\{v,w\}$, and the profile $P=(\varphi)$. applying postulate IC_2, we have that $[\Delta^s_\mu(P)]=[\varphi\wedge\mu]=\{v\}$. This implies that $v\in\mathrm{argmax}_{u\in[\mu]}s(P,u)$ and $w\notin\mathrm{argmax}_{u\in[\mu]}s(P,u)$, which leads to $s(\varphi,v)>s(\varphi,w)$. This, in turn, implies that $s(v,v)>s(v,w)$.

For property S_2, suppose there exist interpretations v and w such that $v\neq w$ and $s(v,v)\leq s(w,v)$. Take, now, a formula φ such that $[\varphi]=\{v,w\}$ and μ as before, with $[\mu]=\{v,w\}$ (see Figure 2). Our assumptions, together with property S_1, proven above, allow us to conclude that:

$$s(v,w)<s(v,v)\leq s(w,v)<s(w,w).$$

In other words, $\max\{s(v,v),s(w,v)\}=s(w,v)$ and $\max\{s(v,w),s(w,w)\}=s(w,w)$, which means that:

$$\max\{s(v,v),s(w,v)\}<\max\{s(v,w),s(w,w)\}.$$

	$[\varphi_1]$ $\{v_1, v_2, \dots\}$	$\dots$ $\dots$	$[\varphi_n]$ $\{v_1, v_2, \dots\}$	$\sum$
v_1	$s(v_1, v_1)$	$\dots$	$s(v_1, v_1)$	$ns(v_1, v_1)$
v_2	$s(v_2, v_2)$	$\dots$	$s(v_2, v_2)$	$ns(v_2, v_2)$
w	$\max_{v\in[\varphi_1]} s(v, w)$	$\dots$	$\max_{v\in[\varphi_n]} s(v, w)$	$\sum_{i=1}^{n} \max_{v\in[\varphi_i]} s(v, w)$

Figure 3: Satisfaction indices when $P = (\varphi_1, \dots, \varphi_n)$, $v_1, v_2 \in [(\bigwedge_{\varphi_i\in P} \varphi_i) \wedge \mu]$ and $w \in [\mu]$ but $w \notin [\bigwedge_{\varphi_i\in P}]$.

But, by postulate IC_2, we have that $[\Delta^s_\mu(P)] = \{v, w\}$ and thus it holds that $\max\{s(v, v), s(w, v)\} = \max\{s(v, w), s(w, w)\}$, which leads to a contradiction, and to the conclusion that property S_2 holds.

Finally, taking φ and μ as in the proof for property S_2, and using the result derived there, we conclude that $\max\{s(v, v), s(w, v)\} = s(v, v)$ and that $\max\{s(v, w), s(w, w)\} = s(w, w)$. Postulate IC_2, now, implies that $s(v, v) = s(w, w)$ and hence property S_3 is satisfied.

("$\Leftarrow$") Conversely, we want to show that if s satisfies properties S_{1-3}, then Δ^s satisfies postulate IC_2. To that end, take a profile $P = (\varphi_1, \dots, \varphi_n)$ and a formula μ such that $(\bigwedge_{\varphi_i\in P} \varphi_i) \wedge \mu$ is consistent. We will prove the claim in two steps. First, we show that for any interpretations $v_1, v_2 \in [(\bigwedge_{\varphi_i\in P} \varphi_i) \wedge \mu]$, we have that $s^\Sigma(P, v_1) = s^\Sigma(P, v_2)$. Then, we show, that if w is an interpretation such that $w \in [\mu]$ but $w \notin [\bigwedge_{\varphi_i\in P} \varphi_i]$, then $s(P, w) < s(P, v_1) = s(P, v_2)$.

Indeed, if $v_1 = v_2$, then the first claim is immediate. If $v_1 \neq v_2$, then we reason as follows. Take a formula $\varphi_i \in P$. Using the fact that $v_1 \in [\varphi_i]$ and property S_2, we get that $s(v_1, v_1) > s(v_j, v_1)$, for any $v_j \in [\varphi_i]$ such that $v_j \neq v_1$. Thus, $s(\varphi_i, v_1) = s(v_1, v_1)$, for any $\varphi_i \in P$, and it follows that $s(P, v_1) = ns(v_1, v_1)$ (see Figure 3). Analogously, we get that $s(\varphi_i, v_2) = s(v_2, v_2)$, for any $\varphi_i \in P$ and $s(P, v_1) = ns(v_2, v_2)$. By property S_3, we have that $s(v_1, v_1) = s(v_2, v_2)$. This, in turn, implies that $s(P, v_1) = s(P, v_2)$.

For the second claim, we have that $s(\varphi_i, w) = \max_{v\in[\varphi_i]} s(v, w)$, for any $\varphi_i \in P$. By property S_1, we have that $\max_{v\in[\varphi_i]} s(v, w) \leq s(v_1, v_1)$ Equality is achieved if $w \in [\varphi_i]$: however, we have assumed that $w \notin [\bigwedge_{\varphi_i\in P}]$, and thus there exists at least one $\varphi_i \in P$ such that $w \notin [\varphi_i]$. In other words, at least one of the inequalities is strict. Hence, when we add up all the satisfaction indices for w, we get that $\sum_{i=1}^{n} \max_{v\in[\varphi_i]} s(v, w) < ns(v_1, v_1)$. In conclusion, $s(P, w) < s(P, v_1) = s(P, v_2)$. □

Theorem 2. *If a satisfaction-based merging operator Δ^s satisfies postulate IC_2, then Δ^s satisfies postulate IC_4 if and only if s also satisfies property S_4 (i.e., is symmetric).*

Proof. Take a merging operator Δ^s that satisfies postulate IC_2. By Theorem 1, this implies that the satisfaction measure s satisfies properties S_{1-3}.

("$\Rightarrow$") Suppose that Δ^s satisfies postulate IC_4 but that s is not symmetric, i.e,. there exist interpretations v_1 and v_2 such that $s(v_1, v_2) \neq s(v_2, v_1)$, Take, then, a profile $P = (\varphi_1, \varphi_2)$, with $[\varphi_1] = \{v_1\}$ and $[\varphi_2] = \{v_2\}$, and a constraint μ such that $[\mu] = \{v_1, v_2\}$. We get that $s(P, v_1) = s(v_1, v_1) + s(v_2, v_1)$ and $s(P, v_2) = s(v_1, v_2) + s(v_2, v_2)$. From property S_3 we have that $s(v_1, v_2) = s(v_2, v_2)$, and by postulate IC_4 we get that $s(P, v_1) = s(P, v_2)$. Thus, $s(v_1, v_2) = s(v_2, v_1)$, which is a contradiction.

("$\Leftarrow$") We assume that s is symmetric and set out to show that Δ^s satisfies postulate IC_4. First of all, notice that if s satisfies property S_4, then properties S_1 and S_2 coincide. Second, we have that s satisfies property S_3, and thus the satisfaction of an interpretation with itself is the same across the entire universe. Let us denote $s(v, v) = k$, for $v \in \mathcal{U}$.

Suppose now that Δ^s does not satisfy postulate IC_4. This implies that there exist two formulas φ and φ', and an interpretation $v^* \in [\varphi]$ such that $s((\varphi, \varphi'), v^*) > s((\varphi, \varphi'), v_j)$, for all $v_j \in [\varphi']$, which is further unpacked as saying that:

$$s(\varphi, v^*) + s(\varphi', v^*) > s(\varphi, v_j) + s(\varphi', v_j), \tag{5}$$

for all $v_j \in [\varphi']$.

Next, we have that $s(\varphi, v^*) = \max_{v_i \in [\varphi]} s(v_i, v^*)$. But, since $v^* \in [\varphi]$ and s satisfies property S_2, we get that $s(\varphi, v^*) = s(v^*, v^*) = k$. Analogously, we have that $s(\varphi', v_j) = s(v_j, v_j)$, for all $v_j \in [\varphi_2]$. Plugging this into Inequality 5 and simplifying, we have that:

$$s(\varphi', v^*) > s(\varphi, v_j),$$

for all $v_j \in [\varphi']$. This means that:

$$\max_{v_i \in [\varphi']} s(v_i, v^*) > \max_{v_i \in [\varphi]} s(v_i, v_j),$$

for all $v_j \in [\varphi']$. Suppose $\max_{v_i \in [\varphi']} s(v_i, v^*) = s(v^{**}, v^*)$, for some $v^{**} \in [\varphi']$. Then we get that:

$$s(v^{**}, v^*) > \max_{v_i \in [\varphi]} s(v_i, v_j),$$

for all $v_j \in [\varphi']$, which implies that:

$$s(v^{**}, v^*) > s(v^*, v^{**}),$$

which is a contradiction, since we have assumed that s is symmetric. $\square$

Corollary 1. *The operators Δ^{hH} and Δ^{hD} satisfy postulates IC_{0-8}.*

Proof. For Δ^{hD}, Proposition 1 gives us that it is equivalent to the distance-based operator Δ^{D}, known to satisfy postulates IC_{0-8} [19, 18].

For operator Δ^{hH}, it already follows from Proposition 2 it satisfies postulates $\mathsf{IC}_{0-1,3,5-8}$. For postulates IC_2 and IC_4, notice that the satisfaction measure s_{hH} satisfies properties S_{1-4}. This implies, by Theorems 1 and 2, that Δ^{hH} satisfies postulates IC_2 and IC_4. $\square$

Theorem 3. *If s is a counting index, the satisfaction-based merging operator Δ^s does not satisfy postulate IC_2.*

Proof. Take a counting index s and the satisfaction-based merging operator Δ^s defined on the basis of it. Assume, towards a contradiction, that Δ^s satisfies postulate IC_2. Let $\sigma\colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ witness that s is a counting index. For the alphabet $\mathcal{A} = \{x, y\}$, consider the profile $P = (\varphi)$, where $\varphi = x \leftrightarrow y$, and the constraint $\mu = x \to y$. We have that $[\varphi] = \{\emptyset, xy\}$, $[\mu] = \{\emptyset, x, xy\}$ and, by postulate IC_2, we get that $[\Delta^s_\mu(P)] = \{\emptyset, xy\}$.

Observe, first, that $s(P, \emptyset) = \sigma(0, 0)$, $s(P, x) = \max\{\sigma(0, 1), \sigma(1, 1)\}$, and $s(P, ab) = \max\{\sigma(0, 2), \sigma(2, 2)\}$. From the facts that $\emptyset \in [\Delta^s_\mu(P)]$ and $xy \in [\Delta^s_\mu(P)]$ but $x \notin [\Delta^s_\mu(P)]$ we infer, respectively, that $s(P, \emptyset) > s(P, x)$ and $s(P, xy) > s(P, x)$. This, in turn, means that:

$$\sigma(0, 0) > \max\{\sigma(0, 1), \sigma(1, 1)\}, \tag{6}$$

$$\max\{\sigma(0, 2), \sigma(2, 2)\} > \max\{\sigma(0, 1), \sigma(1, 1)\}. \tag{7}$$

Take, now, $P' = (\varphi')$, where $\varphi' = x \to y$. By postulate IC_2 we get that $[\Delta^s_\mu(P')] = [\varphi' \wedge \mu] = \{\emptyset, x, xy\}$. Observe now that $s(P', \emptyset) = \sigma(0, 0)$, $s(P', a) = \max\{\sigma(0, 1), \sigma(1, 1)\}$ and $s(P', xy) = \max\{\sigma(0, 2), \sigma(1, 2), \sigma(2, 2)\}$. With inequation (6) this implies that:

$$s(P', \emptyset) > s(P', x).$$

Using inequation (7), we have that:

$$\begin{aligned} \max\{\sigma(0, 2), \sigma(1, 2), \sigma(2, 2)\} &\geq \max\{\sigma(0, 2), \sigma(2, 2)\} \\ &> \max\{\sigma(0, 1), \sigma(1, 1)\}, \end{aligned}$$

which implies that:

$$s(P', xy) > s(P', x).$$

Hence $x \notin [\Delta^s_\mu(P')]$, which contradicts the fact, derived by using postulate IC_2, that $[\Delta^s_\mu(P')] = \{\emptyset, x, xy\}$. □

Theorem 4. *If Δ^s is a satisfaction-based merging operator such that s is a counting index with σ as witness, extends* PAV *and satisfies postulate IC_4, then $\sigma(x, y) = 2h(x) - h(y)$, for any $x, y \in \mathbb{R}$.*

Proof. For any sets of atoms $X \subseteq Y \subseteq \mathcal{A}$ with $|X| = x$ and $|Y| = y$, consider the profile $P = (X, Y)$, i.e., a profile consisting of two complete formulas whose models consist only of the interpretations X and Y, respectively. Take, then, a constraint μ such that $[\mu] = \{X, Y\}$. Using postulate IC_4, we obtain that $[\Delta^s_\mu(P)] = \{X, Y\}$ and hence $\sigma(x, x) + \sigma(x, x) = \sigma(x, y) + \sigma(y, y)$. We obtain that:

$$\sigma(x, y) = 2\sigma(x, x) - \sigma(y, y), \quad \text{for all } x, y. \tag{8}$$

Since Δ^s extends PAV, it has to hold that $\sigma(x + 1, y) - \sigma(x, y) = \frac{1}{x+1}$. By induction, there exist functions c and d such that:

$$\sigma(x, y) = c(y) \cdot h(x) + d(y), \quad \text{for all } x, y. \tag{9}$$

By Equation (8) we see that $\sigma(0,y) = -\sigma(y,y)$. Furthermore, by Equation (9), it holds that $\sigma(0,y) = d(y)$ and $\sigma(y,y) = c(y)\cdot h(y) + d(y)$. Hence, we have that:

$$d(y) = -\frac{1}{2}c(y)\cdot h(y),$$

and consequently:

$$\sigma(x,y) = c(y)(2h(x) - h(y)). \tag{10}$$

Let us now prove that $c(x) = c(y)$, for all x, y. For any $X, Y \subseteq \mathcal{A}$ with $|X| = x$, $|Y| = y$ and $|X \cap Y| = 1$, consider the profile $P = (X, Y)$. Take a constraint μ such that $[\mu] = \{X, Y\}$. By postulate IC_4, we obtain that $[\Delta^s_\mu(P)] = \{X, Y\}$, and hence:

$$\sigma(x,x) + \sigma(1,x) = \sigma(y,y) + \sigma(1,y).$$

By Equation (10), we have that:

$$c(x)\big(2h(x) - h(x) + 2 - h(x)\big) = c(y)\big(2h(y) - h(y) + 2 - h(y)\big),$$

i.e., $c(x) = c(y)$. We conclude that the factor $c(x)$ is a multiplicative constant and thus can be ignored. □

Theorem 5. *If all models of the constraint μ have some fixed size k, then the approval-based merging operators Δ^{AV}, Δ^{PAV} and Δ^{bPAV} satisfy all postulates IC_{0-8}.*

Proof. It already follows from Proposition 2 that the operators Δ^s, for $s \in \{\text{AV}, \text{PAV}, \text{bPAV}\}$ satisfy postulates $\mathsf{IC}_{0-1,3,5-8}$ when the models of μ have fixed size k. All that is left to show is that these operators also satisfy postulates IC_2 and IC_4. The simplest way to see this is to notice that if we restrict the satisfaction function to take only interpretations of fixed size k in the second position, then the satisfaction functions s, for $s \in \{\text{AV}, \text{PAV}, \text{bPAV}\}$, satisfy properties S_{1-4}. Therefore, by Theorems 1 and 2, they also satisfy postulates IC_2 and IC_4. □

Proofs of Section 6

Theorem 6. *The merging operators Δ^{PAV} and Δ^{bPAV} satisfy postulate $\mathsf{IC}_{\mathsf{cp}}$, Δ^{hH} satisfies postulate $\mathsf{IC}_{\mathsf{bp}}$, while $\Delta^{\text{H},\Sigma}$, $\Delta^{\text{H,gmax}}$, Δ^{hD} and Δ^{AV} satisfy neither $\mathsf{IC}_{\mathsf{cp}}$ nor $\mathsf{IC}_{\mathsf{bp}}$.*

Proof. The proportionality of Δ^{PAV} and Δ^{bPAV} follows from more general proportionality statements about PAV, e.g., that PAV satisfies D'Hondt proportionality (see, e.g., [5]). For the sake of completeness and readability, we directly prove that $\mathsf{IC}_{\mathsf{cp}}$ is satisfied.

Assume for the sake of a contradiction that the merging operator Δ^{PAV} does not satisfy $\mathsf{IC}_{\mathsf{cp}}$. Then there is a $k \in \{1, \ldots, m\}$ and $w \in [\Delta^{\text{PAV}}_{\mu_k}(P)]$ with an

l-simple profile $P = (v_1^{p_1}, \ldots, v_l^{p_l})$ that is k-integral such that there is a v_x with $|v_x \cap w| \neq \frac{k \cdot p_x}{n}$. First, some observations. It holds that:

$$\sum_{1 \leq i \leq l} \frac{k \cdot p_i}{n} = k,$$

since:

$$\sum_{1 \leq i \leq l} p_i = n.$$

Further, it holds that:

$$\sum_{1 \leq i \leq l} |v_i \cap w| \leq k,$$

since w assigns at most k many atoms to true, and each complete formula assigns disjoint variables to true. Without loss of generality we can assume that the union of all v_i's cover all atoms: otherwise, if there are uncovered atoms, these are not constrained and in the non-trivial case of $k < n$ one can always assign those atoms to false without lowering satisfaction.

This implies that:

$$\sum_{1 \leq i \leq l} |v_i \cap w| = k.$$

Because of these observations, we can infer that:

$$\exists i \text{ s.t. } \frac{k \cdot p_i}{n} < |v_i \cap w| \text{ implies } \exists j \text{ s.t. } \frac{k \cdot p_j}{n} > |v_j \cap w|$$

since the sums of terms $\frac{k \cdot p_i}{n}$ and $|v_i \cap w|$ both add up to k, and it cannot be the case that if one is strictly smaller that all others are smaller or equal. Likewise, we get that:

$$\exists i \text{ s.t. } \frac{k \cdot p_i}{n} > |v_i \cap w| \text{ implies } \exists j \text{ s.t. } \frac{k \cdot p_j}{n} < |v_j \cap w|.$$

By assumption, we have $|v_x \cap w| \neq \frac{k \cdot p_x}{n}$. This implies that there is an i and j such that:

$$|v_i \cap w| > \frac{k \cdot p_i}{n},$$

and:

$$|v_j \cap w| < \frac{k \cdot p_j}{n}.$$

Consider now an interpretation w' that is equal to w, except for the fact that one atom assigned true in v_i is switched to false and one atom assigned false in v_j is assigned true (i.e., we satisfy i less and j more with still having k many atoms assigned to true). The interpretation w' has a higher score than w if $s^{\text{PAV}}(P, w') > s^{\text{PAV}}(P, w)$. Since satisfaction for all agents except i and j remain the same (recall disjointedness of formulas), we get that satisfaction increases if

$p_i \cdot h(|v_i \cap w'|) + p_j \cdot h(|v_j \cap w'|) > p_i \cdot h(|v_i \cap w|) + p_j \cdot h(|v_j \cap w|)$. These terms are, in fact, close: $h(|v_i \cap w'| + 1) = h(|v_i \cap w|)$ and $h(|v_j \cap w'| - 1) = h(|v_j \cap w|)$. That means we get increased satisfaction if:

$$p_i \cdot \frac{1}{|v_i \cap w|} < p_j \cdot \frac{1}{|v_j \cap w| + 1},$$

since we lessen the satisfaction for each agents i by $\frac{1}{|v_i \cap w|}$ (the last term of the calculcation of the harmonic number) and increases for each agent in j by $\frac{1}{|v_j \cap w| + 1}$ (term added via the harmonic function). By assumption, we have:

$$p_i \cdot \frac{1}{|v_i \cap w|} < p_i \cdot \frac{1}{\frac{p_i \cdot k}{n} + 1},$$

since $|v_i \cap w| > \frac{p_i \cdot k}{n}$ and, by assumption of each of these being an integer, we get $|v_i \cap w| \geq \frac{p_i \cdot k}{n} + 1$, and, in turn, the denominator can only decrease (or stay the same). Likewise, we get:

$$p_j \cdot \frac{1}{|v_j \cap w| + 1} > p_j \cdot \frac{1}{\frac{p_j \cdot k}{n} - 1 + 1}.$$

In turn, satisfaction increases if the following inequality holds:

$$p_i \cdot \frac{1}{\frac{p_i \cdot k}{n} + 1} < p_j \cdot \frac{1}{\frac{p_j \cdot k}{n} - 1 + 1}.$$

Equivalently, if the following inequality holds:

$$p_i \cdot \frac{n}{p_i \cdot k + n} < \frac{n}{k}.$$

This holds because $p_i \geq 0$ and $k, n > 0$.

The same proof can be used for Δ^{bPAV}, since when fixing models of the integrity constraint to have the same cardinality, reasoning under PAV and bPAV coincides.

For Δ^{hH}, an analogous proof can be applied. Assume Δ^{hH} does not satisfy $\mathsf{IC}_{\mathsf{bp}}$. Then there is a 2-simple profile $P = (v_1^{p_1}, v_2^{p_2})$ that is m-integral and an interpretation $w^* \in [\Delta_\mu^{\mathrm{hH}}(P)]$ such that:

$$m - d_{\mathrm{H}}(v_i, w^*) = \frac{m \cdot p_i}{n} \text{ for } i \in \{1, 2\}, \tag{11}$$

but Equality (1) does not hold for all models of $\Delta_\mu(P)$. In particular, this means that there is an interpretation $w \in [\Delta_\mu(P)]$ such that there is a v_x with $|v_x \cap w| \neq \frac{m \cdot p_x}{n}$. Similarly as above, it holds that:

$$\frac{m \cdot p_1}{n} + \frac{m \cdot p_2}{n} = m.$$

By definition, it holds that v_1 and v_2 cover all atoms m. It holds that:

$$\begin{aligned} m - d_{\mathrm{H}}(v_1, w) + m - d_{\mathrm{H}}(v_2, w) &= 2m - \\ &\quad (d_{\mathrm{H}}(v_1, w) + d_{\mathrm{H}}(v_2, w)) \\ &= m, \end{aligned}$$

since if an atom is assigned true it contributes exactly to one of the distances, similarly for assignments to false. As above, we get that:

$$\frac{m \cdot p_1}{n} < m - d_{\mathrm{H}}(v_1, w)$$

implies that:

$$\frac{m \cdot p_2}{n} > m - d_{\mathrm{H}}(v_2, w),$$

and vice versa. The remaining proof is the same as above: just replace $|v_y \cap w|$ with $m - d_{\mathrm{H}}(v_y, w)$, and consider a w' that increases satisfaction of j by one and decreases satisfaction of i by one, by simply switching the truth value of one atom that j is not fully satisfied with. Note that such a w' exists since w^* increases satisfaction of j and decreases satisfaction of i by at least one each (i.e., one can direct the change of w' by w^*). If $w' \notin [\mu]$, then apply the same reasoning iteratively, until w^* is reached (at each step either we arrive at w^* satisfying with equality, or have the same assumption that j is less satisfied and i is more satisfied than required by proportionality).

What is left to be shown is that $\Delta^{\mathrm{H},\Sigma}$, $\Delta^{\mathrm{H},\mathrm{gmax}}$ and Δ^{AV} do not satisfy either of postulates $\mathsf{IC}_{\mathsf{cp}}$ and $\mathsf{IC}_{\mathsf{bp}}$. For that, take $\mathcal{A}$, P and μ as in Example 6. It is easy to see that P is a 2-simple profile. In fact, we can write $P = (v_1^4, v_2^1)$, where $v_1 = x_1x_2x_3x_4x_5$ and $v_2 = y_1y_2y_3y_4y_5$. It is, moreover, straightforward to check that P is 5-integral and 10-integral (10 is the number of atoms in $\mathcal{A}$). We can now ask what satisfaction of $\mathsf{IC}_{\mathsf{cp}}$ and $\mathsf{IC}_{\mathsf{bp}}$ would entail. Given a merging operator Δ that satisfies $\mathsf{IC}_{\mathsf{cp}}$ and an interpretation $w \in [\Delta_\mu(P)]$, postulate $\mathsf{IC}_{\mathsf{cp}}$ requires that:

$$\begin{aligned} |v_1 \cap w| &= \frac{5 \cdot 4}{5} \\ &= 4, \end{aligned}$$

and:

$$\begin{aligned} |v_2 \cap w| &= \frac{5 \cdot 1}{5} \\ &= 1. \end{aligned}$$

In other words, an optimal outcome needs to contain four atoms from X and one from Y. Such an outcome is, for instance, $x_1x_2x_3x_4y_1$.

Similarly, given a merging operator Δ that satisfies $\mathsf{IC}_{\mathsf{bp}}$ and an interpretation $w \in [\Delta_\mu(P)]$, postulate $\mathsf{IC}_{\mathsf{bp}}$ requires that if there is an interpretation w such that:

$$10 - d_{\mathrm{H}}(v_1, w) = \frac{10 \cdot 4}{5}$$

$$= 8,$$

and:

$$10 - d_{\mathrm{H}}(v_2, w) = \frac{10 \cdot 1}{5}$$
$$= 2,$$

then these equalities hold for any other interpretation in $[\Delta_\mu(P)]$. One interpretation in $[\mu_4]$ that satisfies these equalities is $x_1x_2x_3x_4y_1$. Now, postulate $\mathsf{IC_{bp}}$ tells us that all models of $\Delta_\mu(P)$ need to be of this type, i.e., contain four atoms from X and one from Y.

Note now that, as Example 6 illustrates, none of the operators $\Delta^{\mathrm{H},\Sigma}$, $\Delta^{\mathrm{H,gmax}}$ and Δ^{AV} selects such interpretations. Thus, none of these operators satisfies either of postulates $\mathsf{IC_{cp}}$ and $\mathsf{IC_{bp}}$. $\square$

Theorem 7. *There is no merging operator that satisfies $\mathsf{IC_1}$ and both $\mathsf{IC_{cp}}$ and $\mathsf{IC_{bp}}$.*

Proof. Assume the contrary, and take the same instance as in Example 5. By $\mathsf{IC_1}$ it follows that at least one interpretation is selected by the presumed merging operator. By $\mathsf{IC_{cp}}$ we get that if an interpretation w is selected, it holds that three atoms out of $\{x_1, \ldots, x_6\}$ are assigned true, and one of $\{x_7, x_8\}$ (otherwise the postulate is violated). Since μ may be chosen arbitrarily for $\mathsf{IC_{bp}}$, and there is a w' with assigning four atoms of $\{x_1, \ldots, x_6\}$ to true that satisfies μ_4 and the remaining conditions of $\mathsf{IC_{bp}}$, it holds that all outcomes of the presumed merging operator satisfy the implied condition of $\mathsf{IC_{bp}}$. Since none of the interpretations assigning three x_i atoms and one y_j atom to true satisfies the conditions prescribed by $\mathsf{IC_{bp}}$, it follows that the presumed merging operator does not exist (since one interpretation of the type of w must be part that violates $\mathsf{IC_{bp}}$). $\square$

Proofs of Section 7

In this section we prove the formal results of Theorem 8. In order to do achieve that, we show a series of intermediate and auxiliary results.

Proposition 5. *The problem of deciding whether a formula is entailed by the result of merging a profile w.r.t. satisfaction-based merging operators Δ^x is*

- *in Θ_2^P for $x = \mathrm{AV}$, and*
- *in Δ_2^P for $x \in \{\mathrm{PAV}, \mathrm{bPAV}, \mathrm{hH}\}$.*

Proof. Let $P = (\varphi_1, \ldots, \varphi_n)$ be a profile over vocabulary $|\mathcal{A}| = m$, μ an integrity constraint (a formula), and $x \in \{\mathrm{AV}, \mathrm{PAV}, \mathrm{bPAV}, \mathrm{hH}\}$. Consider the following non-deterministic algorithm. Non-deterministically construct $n+1$ interpretations $v_1, \ldots, v_n$, and w. Check, in polynomial time, whether $w \in [\mu]$ and $v_i \in [\varphi_i]$ for $1 \le i \le n$. Compute $z = \sum_{1 \le i \le n} s_x(v_i, w)$. This computation is feasible in polynomial time: computing (cardinalities of) intersections is achievable in polynomial time, and applying the harmonic function to natural numbers is achievable in polynomial time whenever its argument is polynomially bounded by the input size (in our case the number is at most the size of the vocabulary of the profile). It holds that $z \leq s_x(P, w)$ (i.e., the satisfaction of the profile regarding w is bounded from below by z). This holds, because $s_x(v_i, w) \leq max\{s_x(v, w) \mid v \in [\varphi_i]\} = s_x(\varphi_i, w)$ (i.e., satisfaction by v_i provides a lower bound to maximum satisfaction). Complete the algorithm by a binary search over the values $V_x(n', m') = \{s_x(P', w') \mid \exists P' = (\varphi'_1, \ldots, \varphi'_{n'})$ s.t. P' and w' are over vocabulary $\mathcal{A}'$ with $|\mathcal{A}'| = m'\}$. That is, $V_x(n', m')$ contains all possible values for satisfaction index x over profiles of size n' and vocabulary of size m'. We approximate $V_x(n, m)$ depending on x.

($x = \text{AV}$) By definition, $s_{\text{AV}}(a, b) = |a \cap b|$. This intersection is lowest if $|a \cap b| = 0$ and largest, for a profile of size n over vocabulary of size m, if $|a \cap b| = m$. Aggregating n values of integers in $[0, m]$ results in integer interval $[0, m * n]$. A binary search over $V_{\text{AV}}(n, m)$ requires $log(V_{\text{AV}}(n, m))$ many oracle calls, and, in turn, logarithmically many calls w.r.t. the input. Thus, it holds that the problem is in Θ_2^P for $x = \text{AV}$.

($x \in \{\text{PAV}, \text{hH}\}$) For both satisfaction indices it holds that the satisfaction is bounded from below by 0 and by $h(a)$ for some $a \leq m$. For PAV, the intersection is at most m, for hH if the Hamming distance is 0 we obtain the maximum of m. In contrast to AV, we have to include certain rational numbers to approximate $V_x(n, m)$.

First, an upper bound for satsfaction of an interpretation regarding a profile is:

$$\underbrace{h(m) + \cdots + h(m)}_{n \text{ times}} \leq n \cdot h(m).$$

For one harmonic number, i.e., for $0 < a \leq m$ we have:

$$h(a) = \sum_{i=1}^{a} \frac{1}{i} = 1 + \frac{1}{2} + \cdots + \frac{1}{a}$$

and in turn:

$$\begin{aligned} h(a) \cdot m! &= m! + \frac{m!}{2} + \cdots + \frac{m!}{a} \\ &\leq m \cdot m! \end{aligned}$$

with $h(a) \cdot m!$ being an integer (each $\frac{m!}{b}$ is an integer since b occurs in the sequence of factors in $m!$). This means we can represent the set $V_x(n, m)$ by integers from 0 to:

$$n \cdot m \cdot m!$$

and in the binary search scheme the number of oracle calls we can consider is:

$$log(n \cdot m \cdot m!) \leq log(n \cdot m \cdot m^m)$$

$$
\begin{aligned}
&\leq \mathit{log}(g \cdot g \cdot g^g) \\
&= \mathit{log}(g^{g+2}) \\
&= (g+2) \cdot \mathit{log}(g) \\
&= \alpha,
\end{aligned}
$$

where $g = \max\{n, m\}$. The last term is polynomial in the input (since g is bounded polynomially by the input). That is, apply binary search by choosing, initially, $\frac{\alpha}{2}$ and check whether there are interpretations $v_1, \ldots, v_n$, and w such that $s_x(P, w) \geq s_x(v_1, w) + \cdots + s_x(v_n, w) = \beta$ and $\beta \cdot m! \geq \frac{\alpha}{2}$. If successful, continue the binary search above $\frac{\alpha}{2}$, otherwise below $\frac{\alpha}{2}$.

(x = bPAV) By definition, we have $s_{\text{bPAV}}(v, w) = 2 \cdot h(|v \cap w|) - h(|w|)$. The lowest value is when the intersection is empty (e.g., $v = \emptyset$) and $|w| = m$ is maximal. Then we have $-h(m)$ as the lowest value. The maximal value is approximated by $2 \cdot h(m)$. This means, we can represent the approximated set $V_{\text{bPAV}}(n, m)$ by the integer set $[-\alpha, (\alpha + \mathit{log}(2))]$. Using the same idea as above for the binary search, we arrive at an algorithm using polynomial many oracle calls. □

For the convenience of the reader, we recall the standard distance-based merging operator $\Delta^{d_{\text{H}},\Sigma}$ defined via symmetric difference (Hamming distance) and summation aggregation function.

Definition 1. *Let $P = (\varphi_1, \ldots, \varphi_n)$ be a profile and μ an integrity constraint. Define distance-based merging operator $\Delta^{\text{H},\Sigma}$ by $[\Delta_\mu^{\text{H},\Sigma}(P)] = \{w \in [\mu] \mid d_{\text{H}}(P, w) \leq d_{\text{H}}(P, w'), \forall w' \in [\mu]\}$, with $d_{\text{H}}(P, w) = \sum_{\varphi \in P} \mathit{min}\{d_{\text{H}}(v, w) \mid v \in [\varphi]\}$ and $d_{\text{H}}(v, w) = |v \triangle w|$.*

A complexity result of this operator has been established already as Proposition 4 in [20], and is reproduced here.

Proposition 6. *Deciding whether a formula is entailed by the result of distance-based merging operator $\Delta^{\text{H},\Sigma}$ for a given profile is Θ_2^P-complete. Hardness holds even in the case when the profile consists of a single formula that consists of a conjunction of atoms.*

The hardness for the fragment of a single formula consisting of a conjunction of atoms in the profile is stated in the proof of [20, Proposition 4].

Lemma 3. *Let $P = (\varphi)$ be a singleton profile over vocabulary $\mathcal{A}$ s.t. $\varphi = \bigwedge_{y \in Y} y$ is a conjunction of positive literals over $Y \subseteq \mathcal{A}$. It holds that $[\Delta_\mu^{\text{H},\Sigma}(P)] = [\Delta_\mu^{\text{AV},\Sigma}(P)]$ for any μ (see Definition 1).*

Proof. For an interpretation $w \in \mathcal{U}$, let v_w be the interpretation defined by $v_w(x) = w(x)$ for $x \in \mathcal{A} \setminus Y$ and $v_w(y)$ is true for any $y \in Y$. It holds that:

$$
\begin{aligned}
w \in [\Delta_\mu^{\text{H},\Sigma}(P)] &\text{ iff } w \in \operatorname{argmin}_{w' \in [\mu]}(d_{\text{H}}(\varphi, w')) \\
&\text{ iff } w \in \operatorname{argmin}_{w' \in [\mu]}(d_{\text{H}}(v_{w'}, w')) \quad (*)
\end{aligned}
$$

$$
\begin{aligned}
&\text{iff } w \in \operatorname{argmin}_{w' \in [\mu]}(|Y \setminus w'|)\\
&\text{iff } w \in \operatorname{argmax}_{w' \in [\mu]}(|Y \cap w'|)\\
&\text{iff } w \in \operatorname{argmax}_{w' \in [\mu]}(|v'_w \cap w'|)\\
&\text{iff } w \in \operatorname{argmax}_{w' \in [\mu]}(s_{\mathrm{AV}}(v_{w'}, w'))\\
&\text{iff } w \in \operatorname{argmax}_{w' \in [\mu]}(s_{\mathrm{AV}}(\varphi, w')) \quad (**)\\
&\text{iff } w \in [\Delta_\mu^{\mathrm{AV}}(P)].
\end{aligned}
$$

It holds that $d_{\mathrm{H}}(\varphi, w')$ is equal to $d_{\mathrm{H}}(v_{w'}, w')$ $(*)$, since one can always choose a model $v_{w'} \in [\varphi]$ that has an equal assignment on all atoms as w', except for atoms in Y, which must be all assigned to true. The same holds for $(**)$. □

The following definition can be used to express when two satisfaction indices behave "the same", in particular, in the case of a single voter. This is useful for transferring complexity known for belief revision to our setting to obtain complexity (hardness) results.

Definition 2. *The satisfaction indices s and s' are order preserving w.r.t. set of interpretations $W \subseteq \mathcal{U}$ if for all $v, w, v', w' \in W$ and all $\circ \in \{<, =, >\}$ we have*

$$s(v, w) \circ s(v', w') \textit{ iff } s'(v, w) \circ s'(v', w').$$

Lemma 4. *Let $P = (\varphi)$ be a singleton profile, and μ an integrity constraint. If s and s' are order preserving for $[\mu]$, then $[\Delta_\mu^{s,\Sigma}(P)] = [\Delta_\mu^{s',\Sigma}(P)]$.*

Proof. It holds that:

$$
\begin{aligned}
w \in [\Delta_\mu^s(P)] \text{ iff } & w \in [\mu], \exists v \in [\varphi] \text{ s.t.} \nexists w' \in ([\mu] \setminus \{w\}),\\
& \quad \nexists v' \in [\varphi] \text{ with } s(v, w) < s(v', w')\\
\text{iff } & w \in [\mu], \exists v \in [\varphi] \text{ such that}\\
& \quad \nexists w' \in ([\mu] \setminus \{w\}), \nexists v' \in [\varphi]\\
& \quad \text{with } s'(v, w) < s'(v', w')\\
\text{iff } & w \in [\Delta_\mu^{s',\Sigma}(P)].
\end{aligned}
$$

□

Lemma 5. *Satisfaction indices s_{AV} and s_{PAV} are order preserving for any $W \subseteq \mathcal{U}$.*

Proof. Recall that $s_{\mathrm{AV}}(v, w) = |v \cap w|$ and that $s_{\mathrm{PAV}}(v, w) = h(|v \cap w|)$. For any $v, v', w, w' \in \mathcal{U}$ it holds that $s_{\mathrm{AV}}(v, w) = s_{\mathrm{AV}}(v', w')$ iff $|v \cap w| = |v' \cap w'|$. Similarly, $h(x) = h(y)$ iff $x = y$, and, thus, $s_{\mathrm{PAV}}(v, w) = s_{\mathrm{PAV}}(v', w')$ iff $|v \cap w| = |v' \cap w'|$. The proof is completed by considering:

$$
\begin{aligned}
s_{\mathrm{AV}}(v, w) < s_{\mathrm{AV}}(v', w') &\text{ iff } |v \cap w| < |v' \cap w'|\\
&\text{ iff } h(|v \cap w|) < h(|v' \cap w'|)\\
&\text{ iff } s_{\mathrm{PAV}}(v, w) < s_{\mathrm{PAV}}(v', w').
\end{aligned}
$$

The reasoning is analogous for $>$. □

Lemma 6. *The satisfaction measures s_{hH} and the one defined as $s(v,w) = m - d_{\mathrm{H}}(v,w)$ are order preserving, for any $W \subseteq \mathcal{U}$.*

Proof. It holds that, for any $v, v', w, w' \in \mathcal{U}$, we have:

$$\begin{aligned} s_{\mathrm{hH}}(v,w) = s_{\mathrm{hH}}(v',w') &\text{ iff } h(m - d_{\mathrm{H}}(v,w)) = \\ &\qquad\qquad h(m - d_{\mathrm{H}}(v',w')) \\ &\text{ iff } m - d_{\mathrm{H}}(v,w) = \\ &\qquad\qquad m - d_{\mathrm{H}}(v',w') \\ &\text{ iff } s(v,w) = s(v',w'). \end{aligned}$$

The proof for $\circ \in \{<,>\}$ is analogous. □

Lemma 7. *Satisfaction indices s_{AV}, s_{PAV}, and s_{bPAV} are order preserving for $W \subseteq \mathcal{U}$ if $\forall w, w' \in W$ we have $|w| = |w|'$.*

Proof. First, recall that s_{AV} and s_{PAV} are order preserving, for any $W' \subseteq \mathcal{U}$ (due to Lemma 5). Let $v, v', w, w' \in W$. By assumption, we have $|v| = |v'| = |w| = |w'|$. It holds that

$$\begin{aligned} s_{\mathrm{PAV}}(v,w) < s_{\mathrm{PAV}}(v',w') &\text{ iff } h(|v \cap w|) < h(|v' \cap w'|) \\ &\text{ iff } 2 \cdot h(|v \cap w|) < 2 \cdot h(|v' \cap w'|) \\ &\text{ iff } 2 \cdot h(|v \cap w|) - h(|w|) < \\ &\qquad 2 \cdot h(|v' \cap w'|) - h(|w'|) \quad (*) \\ &\text{ iff } s_{\mathrm{bPAV}}(v,w) < s_{\mathrm{bPAV}}(v',w'). \end{aligned}$$

Note that $h(|w|) = h(|w'|)$, due to presumption of same cardinality of interpretations in W $(*)$. For $\circ \in \{=,>\}$, the proof is analogous. □

Proposition 7. *Deciding whether a formula is entailed by the merging result of merging operator Δ^{bPAV} is Θ_2^P hard. Hardness holds even when the profile is a singleton and all models of the integrity constraint have the same cardinality.*

Proof. We show a reduction from the Θ_2^P hard problem of UCOSAT that is an adaptation of [7, Theorem 6.9]. Let an instance of UCOSAT be given by the formula $\psi = \{C_1, \ldots, C_\ell\}$ with C_i propositional clauses, and V_ψ be the propositional variables of ψ. That is, ψ is in conjunctive normal form. An instance is a "yes" instance iff it holds that each satisfiable set $S \subseteq \psi$ of maximum cardinality satisfies exactly the same clauses. That is, the problem asks whether each satisfiable set of clauses that has maximum cardinality satisfies the same clauses.

We construct a merging instance with a single knowledge base as follows. Let $V' = \{z, c_1, \ldots, c_\ell, c'_1, \ldots, c'_\ell\}$ and $\overline{V'} = \{\overline{c_1}, \ldots, \overline{c_\ell}, \overline{c'_1}, \ldots, \overline{c'_\ell}\}$ be two sets of new variables such that $V_\psi \cap V' = V_\psi \cap \overline{V'} = \emptyset$. Further, let $\overline{V_\psi} = \{\overline{x} \mid x \in V_\psi\}$. As in the proof of [7, Theorem 6.9] we define $T_1 = \{c_1, \ldots, c_\ell\}$, $p_1 = (C_1 \vee \neg c_1) \wedge \cdots \wedge (C_\ell \vee \neg c_\ell)$, $T'_1 = \{c'_1, \ldots, c'_\ell\}$, and $p'_1 = (C'_1 \vee \neg c'_1) \wedge$

$\cdots \wedge (C'_\ell \vee \neg c'_\ell)$. We set $\mathcal{A} = V_\psi \cup \overline{V_\psi} \cup V' \cup \overline{V'}$, $P = (\varphi)$, $\varphi = T_1 \wedge T'_1$, and $\mu = p_1 \wedge p'_1 \wedge \varphi_D \wedge z \leftrightarrow ((c_1 \leftrightarrow c'_1) \wedge \cdots \wedge (c_\ell \leftrightarrow c'_\ell))$, where $\varphi_D = \{(x \leftrightarrow \neg\overline{x}) \mid x \in V_\psi \cup V'\}$. Observe that μ, more specifically the subformula φ_D, enforces that for all solutions $w \in \Delta^{\mathrm{bPAV},\Sigma}_\mu(P)$, $|w| = \frac{|\mathcal{U}|}{2}$. As a consequence, the term $h(|w|)$ of the s_{bPAV} satisfaction index becomes constant for all such w, and hence $s_{\mathrm{bPAV}}(v, w)$ is order preserving w.r.t. $s_{\mathrm{AV}}(v, w)$. Nevertheless, the Hamming distance between $[T_1]$ and w (as well as $[T'_1]$ and w) remains the same. By Lemma 3, it follows that computing the merging results with respect to Hamming distance and sum aggregation function coincides with the result by the satisfaction-based merging operator using AV. It remains to show that $\Delta^{\mathrm{bPAV},\Sigma}_\mu(P) \models z$ iff ψ is a yes-instance of UOCSat, which can be done analogously to the original proof [7, Theorem 6.9]. $\square$

Corollary 2. *Deciding whether a formula is entailed by the merging result of merging operators Δ^s_μ for $s \in \{\mathrm{AV}, \mathrm{PAV}, \mathrm{bPAV}, \mathrm{hH}\}$ is Θ^P_2 hard.*

For the complexity of the standard merging operators $\Delta^{\mathrm{H,gmax}}$ and Δ^{D}, we will introduce a new notion and an intermediary result. A *knowledge-base* K is a set of propositional formulas. The set of models of K is defined as $[K] = \bigcap_{\varphi \in K}[\varphi]$. We say that K is consistent if $[K] \neq \emptyset$. We recall now that, given a pseudo-distance d, a formula φ and an interpretation w, the distance $d(\varphi, w)$ between φ and w is defined as $d(\varphi, w) = \min_{v \in [\varphi]} d(v, w)$.

Lemma 8. *If K is a consistent knowledge-base, d is a pseudo-distance and w is an interpretation, then it holds that:*

$$\max_{\varphi \in K} d(\varphi, w) = d(\bigwedge_{\varphi \in K} \varphi, w).$$

In other words, the maximal distance between formulas in K and w is the same as the distance between the formula $\bigwedge_{\varphi \in K}$ and w. Lemma 8 allows us to identify the operators $\Delta^{\mathrm{H,gmax}}$ and Δ^{D} with the operators $\Delta^{\mathrm{H,max,lex}}$ and $\Delta^{\mathrm{D,max},\Sigma}$ from [20], where the former operator was shown to be Δ^P_2-complete, while the latter operator was shown to be Θ^P_2-complete. This justifies the entries in Table 3.

C Thoughts on the Bounded PAV operator

The satisfaction index s_{bPAV} (and in consequence also the merging operator Δ^{bPAV}) exhibit an interesting behavior: In general it is possible that an interpretation gets a negative score assigned by s_{bPAV} (w.r.t. some w). As the size of w increases, the larger the total "penalty" becomes. The penalty added for increasing the size of $|w|$ by one, however, decreases according to the harmonic function. Such a behavior is intended if increasing the size of, e.g., a committee, is costly but eventually there is an amortization of the size. Thus, the merging operator needs to find a balance between the cost for the committee and preferences of minorities.

	$[\varphi_1] = [\varphi_2] = [\varphi_3]$ $\{a\}$	$[\varphi_4]$ $\{b\}$	$\sum$
$\emptyset$	$3 \cdot \quad 0$	0	0
a	$3 \cdot \quad 1$	-1	2
b	$3 \cdot \quad -1$	1	-2
ab	$3 \cdot \quad 0.5$	0.5	2

Table 8: Satisfaction indices with respect to bPAV when $P = (\varphi_1, \ldots, \varphi_4)$, $[\varphi_1] = [\varphi_2] = [\varphi_3] = \{a\}$, $[\varphi_4] = \{b\}$, and $\mu = \top$.

An example where such a behavior might be desirable is the process of finding new hardware for the institute. In such a setting usually not all details regarding the hardware configurations are known upfront, but they are only known after a concrete offer has been received from the supplier. So the candidates represent different possible hardware configurations (without knowing all details) and the members of the institute reveal their (a-priori) preferences regarding this options. As requesting a concrete offer from the suppliers can be time-consuming the technician only wants to request an offer if there is a significant support for the respective alternative. Hence, adding another alternative to the list of possible options must have a large enough support from the members. Still, if requesting a high number of offers is necessary at some point the additional cost for each offer usually show effects of amortization.

Furthermore, we want to highlight that the empty solution $w = \emptyset$ trivially has $s_{\mathrm{bPAV}}(v, w) = 0$ for any $v \in \mathcal{U}$. Thus, the empty solution gives us a natural baseline to assess the quality of other interpretations. Notice that this immediately rules out all solutions with negative score as they are trivially dominated by the empty solution.

Next we present an example for this merging operator in Table 8. As already discussed above the empty committee indeed scores the baseline of 0. The interpretation b yields a negative score of -2 caused by the fact that b has only very weak support (namely by φ_4) but a cardinality of one resulting in a quite large penalty, and hence this solution is always withdrawn. Interestingly both interpretations a and ab are tied with a score of 2. This is because there is a balance between the penalty for adding another candidate to the solution and the gain for satisfying φ_4 as well.

In the next two results we describe this balance more in detail. Thereby, as a side result, we also establish an relation between s_{bPAV} and s_{PAV}. In a first step we characterize the gain for increasing the committee size by one for an arbitrary interpretation v. After that we move on to lift this statement to arbitrary profiles.

Lemma 9. *Let v, w, and w' be interpretations such that $|w| + 1 = |w'|$. Then the gain of v with respect to s_{bPAV} for increasing the size of the committee by one is positive if $s_{\mathrm{PAV}}(v, w') - s_{\mathrm{PAV}}(v, w) > \frac{1}{2(k+1)}$.*

Proof. This can be seen by transforming the difference of the scores for w and w', i.e., $s_{\text{bPAV}}(v,w') - s_{\text{bPAV}}(v,w)$, as follows:

$$
\begin{aligned}
s_{\text{bPAV}}(v,w') - s_{\text{bPAV}}(v,w) &= (2h(|v \cap w'|) - h(w')) \\
&\quad - (2h(|v \cap w|) - h(|w|)) \\
&= (2h(|v \cap w'|) - 2h(|v \cap w|)) \\
&\quad - h(w') + h(|w|) \\
&= 2\left(h(|v \cap w'|) - h(|v \cap w|)\right) \\
&\quad - \frac{1}{k+1}
\end{aligned}
$$

Thus the gain *for* v is positive if:

$$h(|v \cap w'|) - h(|v \cap w|) > \frac{1}{2(k+1)},$$

or, equivalently:

$$s_{\text{PAV}}(v,w') - s_{\text{PAV}}(v,w) > \frac{1}{2(k+1)},$$

which concludes the proof. □

Observe that this result indeed relates the notions of s_{bPAV} to s_{PAV}. In a next step we lift the result to arbitrary profiles.

Lemma 10. *Let P be a profile, and w, w' be interpretations such that $|w|+1 = |w'|$. Then the total gain of P with respect to s_{bPAV} for increasing the size of the committee by one is positive if $s_{\text{PAV}}(v,w') - s_{\text{PAV}}(v,w) > \frac{1}{2(k+1)}$.*

Proof. In this proof we directly use the definition of s_{bPAV} and s_{PAV} to obtain the desired inequality. It holds that:

$$\sum_{\varphi \in P} s_{\text{bPAV}}(\varphi, w') - \sum_{\varphi \in P} s_{\text{bPAV}}(\varphi, w)$$

is equal to:

$$\sum_{\varphi \in P} (2s_{\text{PAV}}(\varphi, w') - h(w')) - \sum_{\varphi \in P} (2s_{\text{PAV}}(\varphi, w) - h(|w|)),$$

which is in turn equal to:

$$2\sum_{\varphi \in P} (s_{\text{PAV}}(\varphi, w') - s_{\text{PAV}}(\varphi, w)) + n\,(-h(w') + h(|w|)).$$

Thus the *total* gain is positive if:

$$\sum_{\varphi \in P} (s_{\text{PAV}}(\varphi, w') - s_{\text{PAV}}(\varphi, w)) > \frac{n}{2(k+1)},$$

which concludes the proof. □