

Strong Equivalence for Argumentation Frameworks with Collective Attacks

Wolfgang Dvořák, Anna Rapberger, and Stefan Woltran

Institute of Logic and Computation, TU Wien, Austria

Abstract. Argumentation frameworks with collective attacks are a prominent extension of Dung’s abstract argumentation frameworks, where an attack can be drawn from a set of arguments to another argument. These frameworks are often abbreviated as SETAFs. Although SETAFs have received increasing interest recently, the notion of strong equivalence, which is fundamental in nonmonotonic formalisms to characterize equivalent replacements, has not yet been investigated. In this paper, we study how strong equivalence between SETAFs can be decided with respect to the most important semantics and also consider variants of strong equivalence.

1 Introduction

Abstract argumentation frameworks (AFs) as introduced by Dung [6] are a core formalism in formal argumentation. A popular line of research investigates extensions of Dung AFs that allow for a richer syntax (see, e.g. [5]). In this work we consider SETAFs as introduced by Nielsen and Parsons [12] which generalize the binary attacks in Dung AFs to collective attacks such that a set of arguments B attacks another argument a but no proper subset of B attacks a . As discussed in [12], there are several scenarios where arguments interact and can constitute an attack on another argument only if these arguments are jointly taken into account. Representing such a situation in Dung AFs often requires additional artificial arguments to “encode” the conjunction of arguments.

SETAFs have received increasing interest in the last years. For instance, semi-stable, stage, ideal, and eager semantics have been adapted to SETAFs in [8, 10]; translations between SETAFs and other abstract argumentation formalisms are studied in [14]; and the expressiveness of SETAFs is investigated in [7]. [17] observed that for particular instantiations, SETAFs provide a more convenient target formalism than Dung AFs.

The notion of strong equivalence is recognized as a central concept in nonmonotonic reasoning [11, 16, 15] and provides means for the replacement property. In terms of AFs, strong equivalence (with respect to a semantics σ) between two frameworks F and G holds, if for any further AF H , $\sigma(F \cup H) = \sigma(G \cup H)$. Hence, replacing a subframework F by a strongly equivalent AF G in any context does not alter the extensions. In other words, the notion of strong

equivalence allows for simplifying a part of an argumentation framework without looking at the rest of the framework; a method that has been recently applied in a practical setting in terms of preprocessing Dung AFs [9].

For Dung AFs, strong equivalence and variants thereof have been extensively studied in the literature [13, 1, 3, 2, 4]. The main results reveal that strong equivalence can be decided by syntactic identity of so-called kernels of the AFs to be compared. In these kernels, depending on the actual semantics, certain inactive attacks need to be removed. Up to date, similar investigations for SETAFs have not been undertaken and it remained open how the concept of inactive attacks can be generalized to the richer attack structure SETAFs provide.

In this paper, we provide characterizations of strong equivalence between SETAFs with respect to admissible, complete, stable, preferred, semi-stable and stage semantics. We do so by generalizing the notion of kernels to SETAFs. Moreover, we show that strong equivalence for the semantics under consideration coincides with weaker notions of equivalence, where we disallow certain context frameworks H in the definition of the equivalence relation. Our results confirm that SETAFs are a natural generalization of AFs in the sense that the appealing concept of kernels also is applicable to SETAFs.

Some proofs are omitted but are provided in a technical report available at www.dbai.tuwien.ac.at/research/report/dbai-tr-2019-116.pdf.

2 Preliminaries

Throughout the paper, we assume a countably infinite domain \mathfrak{A} of possible arguments.

Definition 1. A SETAF is a pair $F = (A, R)$ where $A \subseteq \mathfrak{A}$ is finite, and $R \subseteq (2^A \setminus \{\emptyset\}) \times A$ is the attack relation. SETAFs (A, R) , where for all $(S, a) \in R$ it holds that $|S| = 1$, amount to (standard Dung) AFs. In that case, we usually write (a, b) to denote the set-attack $(\{a\}, b)$. Moreover, for a SETAF $F = (B, S)$, we use $A(F)$ and $R(F)$ to identify its arguments B and respectively its attack relation S .

Given a SETAF (A, R) , we write $S \mapsto_R b$ if there is a set $S' \subseteq S$ with $(S', b) \in R$. Moreover, we write $S' \mapsto_R S$ if $S' \mapsto_R b$ for some $b \in S$. We drop subscript R in \mapsto_R if there is no ambiguity. For $S \subseteq A$, we use S_R^+ to denote the set $\{b \mid S \mapsto_R b\}$ and define the range of S (w.r.t. R), denoted S_R^\oplus , as the set $S \cup S_R^+$.

The notions of conflict and defense naturally generalize to SETAFs.

Definition 2. Given a SETAF $F = (A, R)$, a set $S \subseteq A$ is conflicting in F if $S \mapsto_R a$ for some $a \in S$. A set $S \subseteq A$ is conflict-free in F , if S is not conflicting in F , i.e. if $S' \cup \{a\} \not\subseteq S$ for each $(S', a) \in R$. $cf(F)$ denotes the set of all conflict-free sets in F .

Definition 3. Given a SETAF $F = (A, R)$, an argument $a \in A$ is defended (in F) by a set $S \subseteq A$ if for each $B \subseteq A$, such that $B \mapsto_R a$, also $S \mapsto_R B$. A set T of arguments is defended (in F) by S if each $a \in T$ is defended by S (in F).

The semantics we study in this work are the admissible, stable, preferred, complete, stage and semi-stable semantics, which we will abbreviate by *adm*, *stb*, *pref*, *com*, *stage* and *sem* respectively [12, 8, 10].

Definition 4. Given a SETAF $F = (A, R)$ and a conflict-free set $S \in cf(F)$. Then,

- $S \in adm(F)$, if S defends itself in F ,
- $S \in stb(F)$, if $S \mapsto a$ for all $a \in A \setminus S$,
- $S \in pref(F)$, if $S \in adm(F)$ and there is no $T \in adm(F)$ s.t. $T \supset S$,
- $S \in com(F)$, if $S \in adm(F)$ and $a \in S$ for all $a \in A$ defended by S ,
- $S \in stage(F)$, if $\nexists T \in cf(F)$ with $T_R^\oplus \supset S_R^\oplus$, and
- $S \in sem(F)$, if $S \in adm(F)$ and $\nexists T \in adm(F)$ s.t. $T_R^\oplus \supset S_R^\oplus$.

The relationship between the semantics has been clarified in [12, 8, 10] and matches with the relations between the semantics for Dung AFs, i.e. for any SETAF F :

$$stb(F) \subseteq sem(F) \subseteq pref(F) \subseteq com(F) \subseteq adm(F) \subseteq cf(F) \quad (1)$$

$$stb(F) \subseteq stage(F) \subseteq cf(F). \quad (2)$$

The following property also carries over from Dung AFs: For any SETAF F , if $stb(F) \neq \emptyset$ then $stb(F) = sem(F) = stage(F)$.

3 Notions of Strong Equivalence and Basic Concepts

We define the notion of strong equivalence for SETAFs along the lines of [13]. Given SETAFs F, G we define the union of F and G as $F \cup G = (A(F) \cup A(G), R(F) \cup R(G))$.

Definition 5. Two SETAFs F and G are strongly equivalent to each other wrt. a semantics σ , in symbols $F \equiv_s^\sigma G$, iff for each SETAF H , $\sigma(F \cup H) = \sigma(G \cup H)$ holds.

By definition, we have that $F \equiv_s^\sigma G$ implies $\sigma(F) = \sigma(G)$, i.e. standard equivalence between F and G wrt. σ . However, no matter which of the considered semantics we choose for σ , the converse direction does not hold in general (cf. [13]).

We consider two weakenings for Definition 5 by restricting the potential context SETAF H . First, we let H to be only an AF instead of a SETAF. We consider this an interesting restriction in the sense of whether an AF is sufficient to reveal the potential difference between the compared SETAFs in terms of strong equivalence. Another weakening has first been proposed in [1] under the name *normal expansion equivalence*. Here the framework H is not allowed to add attacks between “existing” arguments (in F or G), and thus better reflects that in dynamic scenarios new arguments may be proposed but the relation between given arguments remains unchanged.

Definition 6. Let F and G be SETAFs and σ be a semantics. Moreover, let $B = A(F) \cup A(G)$. We write

- $F \equiv_n^\sigma G$, iff for each SETAF H with $R(H) \cap (2^B \times B) = \emptyset$, $\sigma(F \cup H) = \sigma(G \cup H)$.
- $F \equiv_{sd}^\sigma G$, iff for each AF H , $\sigma(F \cup H) = \sigma(G \cup H)$.

- $F \equiv_{nd}^\sigma G$, iff for each AF H with $R(H) \cap (B \times B) = \emptyset$, $\sigma(F \cup H) = \sigma(G \cup H)$.

Results for strong equivalence between AFs (i.e. \equiv_{sd}^σ in our notation) rely on so-called kernels that remove attacks that do not contribute to the computation of the extensions of an AF F , no matter how the AF is extended to $F \cup H$. This is best illustrated in terms of stable semantics. Consider an attack (a, b) where a is self-attacking. Then, removing (a, b) from the attacks has no effect since (i) the conflicts remain the same (note that a is never part of a conflict-free set, due to the self-attack), (ii) if b needs to be attacked by a stable extension, this cannot happen due to attack (a, b) (again, since a will never part of a stable extension due to its conflict). In [13], it has been shown that removal of such *inactive attacks* is sufficient to decide strong equivalence w.r.t. stable semantics: given an AF F , define its stable kernel as

$$F^{sk} = (A(F), R(F) \setminus \{(a, b) \in R(F) \mid a \neq b, (a, a) \in R(F)\}).$$

For AFs F, G it holds that $F \equiv_{sd}^\sigma G$ iff $F^{sk} = G^{sk}$. For other semantics, the notion of kernel needs to be further restricted; intuitively, an attack (a, b) with self-attacking a might still be responsible for defending b against a .

However, as we will show in the next section, kernels can be defined for SETAFs as well. Before doing so, we first consider the concept of redundant attack and show that they are also redundant when testing for strong equivalence. Then, we generalize the concept of inactive attacks to SETAFs.

Definition 7. Let $F = (A, R)$ be a SETAF. An attack $(S, a) \in R$ is called *redundant in F* if there exists $(S', a) \in R$ with $S' \subset S$.

As shown in [14] we can remove redundant attacks of SETAF F without changing its semantics. When removing all redundant attacks from F the resulting SETAF G is called *minimal form of F* .

Lemma 1. For a SETAF F and its minimal form G we have $F \equiv_s^\sigma G$ for $\sigma \in \{adm, stb, pref, com, stage, sem\}$.

Proof. Let R be the set of redundant attacks in F and consider an arbitrary SETAF H . The attacks in R are also redundant in the SETAF $F \cup H$ and thus, by [14]¹, $\sigma(F \cup H) = \sigma((F \cup H) \setminus R) = \sigma(G \cup H)$. Now as $\sigma(F \cup H) = \sigma(G \cup H)$ for each SETAF H we obtain that $F \equiv_s^\sigma G$. □

We have that two SETAFs are strongly equivalent iff their minimal forms are strongly equivalent. Thus in the remainder of the paper we will assume that SETAFs tested for strong equivalence are in minimal form, i.e. have no redundant attacks.

A crucial role in the definition of kernels plays the concept of an inactive attack which we define right now.

Definition 8. Let $F = (A, R)$ be a SETAF. An attack $(S, a) \in R$ is called *inactive in F* if (i) $a \notin S$ and there exist $S' \subseteq S$ and $b \in S$ such that $(S', b) \in R$, or (ii) $a \in S$ and there exist $S' \subset S$ and $b \in S$ such that $(S', b) \in R$. An attack that is not inactive in F is said to be *active in F* .

¹*sem* and *stage* are not considered in [14] but the result immediately extends to those semantics.

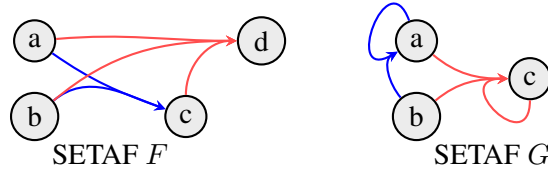


Figure 1: A SETAF F with active attack $(\{a, b\}, c)$ and inactive attack $(\{a, b, c\}, d)$ and a SETAF G with active self-attack $(\{a, b\}, a)$ and inactive self-attack $(\{a, b, c\}, c)$.

Figure 1 illustrates the different notions of inactive attacks. An example of an inactive attack according to case (i) in Definition 8 is given by the SETAF F ; here, the source-set $S = \{a, b, c\}$ is conflicting and attacks an argument $d \notin S$. Case (ii) covers inactive self-attacks; note that in this case, the set S' is required to be a proper subset of S . This subtle difference ensures the existence of active self-attacks since the source-set of each self-attack is conflicting by definition. The SETAF G in Figure 1 provides an example; here, the active self-attack $(\{a, b\}, a)$ deactivates the self-attack $(\{a, b, c\}, c)$. Note that in terms of AFs Definition 8 boils down to the situation discussed above for binary attacks (a, b) with $a \neq b$: $(\{a\}, b)$ is inactive iff a attacks itself. We conclude this section with a technical result.

Lemma 2. *Let $F = (A, R)$ be a SETAF and $(S, a) \in R$ be inactive in F . Then there exists an attack $(S', b) \in R$ with $S' \subseteq S$ and $b \in S$ that is active in F .*

Proof. Towards a contradiction let $(S, a) \in R$ be an attack violating the condition of the lemma such that all inactive attacks $(T, b) \in R$ with $|S \cup \{a\}| > |T \cup \{b\}|$ satisfy the condition, i.e. (S, a) is minimal in this respect. By inactivity of (S, a) either (i) $a \notin S$ and there exist $S' \subseteq S$ and $b \in S$ such that $(S', b) \in R$, or (ii) $a \in S$ and there exist $S' \subset S$ and $b \in S$ such that $(S', b) \in R$. By assumption, (S', b) is inactive in F .

In case (i) we have $|S' \cup \{b\}| < |S \cup \{a\}|$, and, by the minimality of (S, a) , we obtain that there is an active attack $(S'', c) \in R$ with $S'' \subseteq S' \subseteq S$ and $c \in S' \subseteq S$.

The same reasoning applies in case (ii) if $|S' \cup \{b\}| < |S \cup \{a\}|$. Thus assume that $|S' \cup \{b\}| = |S \cup \{a\}|$, i.e. $S' = S \setminus \{b\}$, $b \in S$. By assumption (S', b) is inactive and thus there exist $S'' \subseteq S'$, $c \in S'$ such that $(S'', c) \in R$. But then $|S'' \cup \{c\}| < |S \cup \{a\}|$ and, by the minimality of (S, a) , we obtain that there is an active attack $(S''', d) \in R$ with $S''' \subseteq S'' \subseteq S$ and $d \in S'' \subseteq S$. \square \square

4 Characterizations of Strong Equivalence

In this section we characterize strong equivalence as well as its variants (cf. Definition 6) for all semantics under consideration by introducing three different kernels for the different semantics. We will show that two SETAFs are strongly equivalent iff they have the same kernel of a particular type. We start with the result for stable and stage semantics. In the corresponding kernel all inactive attacks have to be removed. For the two remaining kernels, the situation is slightly different and towards our results for admissible, semi-stable, preferred, and complete semantics we will introduce an additional normal form for SETAFs to handle this situation.

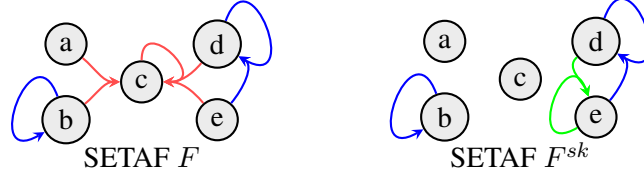


Figure 2: Example illustrating the stable kernel of a SETAF F . Active attacks in blue; inactive in red. Newly introduced self-attacks appear in green.

4.1 Stable Kernel

The main idea of the stable kernel is that for stable semantics only active attacks are relevant. However, for self-attacks (i.e. attacks (S, a) such that $a \in S$) additional care is needed, since self-attacks (S, a) and (S, b) turn out to be indistinguishable. This is due to the fact that self-attacks never contribute to the range of a stable extension and thus only the information that the set S is conflicting is relevant. For example consider the SETAFs $F = (\{a, b\}, \{\{\{a, b\}, a\}\})$ and $G = (\{a, b\}, \{\{\{a, b\}, b\}\})$. The two SETAFs have different active attacks but as we argue next, $F \equiv_s^{stb} G$ holds. Let H be an arbitrary SETAF and let $S \in stb(F \cup H)$. Then S cannot contain both a and b . If $a, b \notin S$, then S attacks (in $F \cup H$) both a and b via attacks in H . Otherwise, wlog let $a \in S$. Then S attacks b via an attack in H . In both cases S is stable in $G \cup H$. That is, for active self-attacks (S, a) only the set S but not the concrete attacked argument $a \in S$ is significant. For conflicting S , we thus add (S, b) for all $b \in S$ to the kernel.

Definition 9. For a SETAF $F = (A, R)$ in minimal form, we define the stable kernel of F as $F^{sk} = (A, R^{sk})$ with

$$R^{sk} = \{(S, a) \in R \mid (S, a) \text{ active in } F\} \cup \{(S, b) \mid (S, a) \text{ active in } F, a \in S, b \in S\}.$$

The stable kernel of an arbitrary SETAF F is the stable kernel of the minimal form of F .

In a first step we show that the stable, and stage respectively, extensions of a SETAF F coincide with the stable, and stage respectively, extensions of its stable kernel F^{sk} . The following result suffices in this endeavor.

Lemma 3. For any SETAF in minimal form F (1) $cf(F) = cf(F^{sk})$ and (2) for each $S \in cf(F)$, $S_{R(F)}^\oplus = S_{R(F^{sk})}^\oplus$.

Proof. (1) $cf(F) \subseteq cf(F^{sk})$: Consider $T \in cf(F)$ and towards a contradiction assume $T \notin cf(F^{sk})$. Thus there is $(S, b) \in R^{sk}$ such that $S \cup \{b\} \subseteq T$. If (S, b) would be an active attack in F then $T \notin cf(F)$ and thus we have that $b \in S$ and there is an $a \in S$ such that $(S, a) \in R(F)$. As $S \subseteq T$ this is in contradiction to our initial assumption $T \in cf(F)$. For $cf(F) \supseteq cf(F^{sk})$, let $T \in cf(F^{sk})$ and (S, a) be any attack in F that is not present in F^{sk} . We have to show that $S \cup \{a\} \not\subseteq T$. From Lemma 2 there exists $(S', b) \in R$ with $S' \subseteq S$ and $b \in S$ that is active in F and thus contained in $R(F^{sk})$. Since $T \in cf(F^{sk})$, $S' \cup \{b\} \not\subseteq T$ and $S \cup \{a\} \not\subseteq T$ follows.

(2) Let $S \in cf(F)$. $S_{R(F)}^\oplus \supseteq S_{R(F^{sk})}^\oplus$: Notice that no attack in the set $\{(S, b) \mid (S, a) \text{ active in } F, a \in S, b \in S\}$ contributes to the range of a conflict-free set and as $R(F) \supseteq R^{sk} \setminus \{(S, b) \mid (S, a) \text{ active in } F, a \in S, b \in S\}$ we obtain that $S_{R(F)}^\oplus \supseteq S_{R(F^{sk})}^\oplus$. For $S_{R(F)}^\oplus \subseteq S_{R(F^{sk})}^\oplus$, let (S', a) be any attack in F that is not present in F^{sk} , i.e. (S', a) is inactive in F . As (S', a) is inactive we have that $S' \not\subseteq S$ and thus the attack does not contribute to the range of S . $\square \square$

Given the above semantical correspondence between SETAFs and their kernels we show that SETAFs with the same kernel are strongly equivalent on a purely syntactic level. That is, we show that if two SETAFs F, G have the same stable kernel then also their expansions with the same SETAF H have the same kernel.

Lemma 4. *Let F and G be SETAFs in minimal form such that $F^{sk} = G^{sk}$. Then, $(F \cup H)^{sk} = (G \cup H)^{sk}$ for all SETAFs H .*

Proof. Notice, that $F \cup H$ (and likewise $G \cup H$) might not be in minimal form. However, by definition, we remove redundant attacks before constructing the kernel. It suffices to show $R((F \cup H)^{sk}) \subseteq R((G \cup H)^{sk})$ as $R((F \cup H)^{sk}) \supseteq R((G \cup H)^{sk})$ then follows by symmetry. Let $(S, a) \in R((F \cup H)^{sk})$. We show that $(S, a) \in R((G \cup H)^{sk})$ by considering two cases.

1) Assume that (S, a) , $a \notin S$, is active and non-redundant in $F \cup H$, i.e. there is no attack $(S', b) \in R(F \cup H)$ such that (i) $S' \subseteq S$, $b \in S \cup \{a\}$ and $(S, a) \neq (S', b)$. We show that (S, a) is active and non-redundant in $G \cup H$, i.e. (a) $(S, a) \in R(G \cup H)$ and (b) there is no attack $(S', b) \in R(I)$, $I \in \{H, G\}$ which satisfies (i). (a) If $(S, a) \in R(H)$, then $(S, a) \in R(G \cup H)$ by definition. Otherwise, if $(S, a) \in R(F)$, then, as the attack is active and non-redundant, we can conclude that $(S, a) \in R(F^{sk}) = R(G^{sk})$ and thus $(S, a) \in G \cup H$. (b) For $R(H)$ this holds by the fact that there is no such attack in $R(F \cup H)$. Notice that as there is no such attack in $R(F)$ there is also no such attack in $R(F^{sk}) = R(G^{sk})$. Towards a contradiction assume that there is an attack $(S', b) \in R(G)$ satisfying (i). Then, by Lemma 2, there is an active attack (T, c) with $T \subseteq S'$ satisfying (i). Thus $(T, c) \in R(G^{sk}) = R(F^{sk})$ and thus $(S, a) \notin R(G^{sk}) = R(F^{sk})$, a contradiction. By (a) and (b) we can conclude that $(S, a) \in R((G \cup H)^{sk})$.

2) Assume that (S, a) is such that $a \in S$ and there is a non-redundant active attack $(S, b) \in R(F \cup H)$ with $b \in S$. If $(S, b) \in R(F)$ then, by the assumption $F^{sk} = G^{sk}$, there is an active and non-redundant attack $(S, c) \in R(G)$ with $c \in S$. Now, as (S, b) is active in $F \cup H$, there is no $(S', d) \in R(H)$ with $S' \subset S$ and $d \in S$ and thus (S, c) is active in $F \cup H$. Hence, $(S, a) \in R((G \cup H)^{sk})$.

Now assume there is no such $(S, b) \in R(F)$. Then $(S, b) \in R(H)$ and thus $(S, b) \in R(G \cup H)$. Towards a contradiction assume (S, b) is redundant or inactive.

- If (S, b) is redundant, i.e. there is $S' \subset S$ with $(S', b) \in R(G \cup H)$ and (S', b) non-redundant. As (S, b) is non-redundant in $F \cup H$ we have that $(S', b) \in R(G)$. If (S', b) is inactive in G then there is an attack (S'', c) with $S'' \cup \{c\} \subseteq S'$ active in G . By $F^{sk} = G^{sk}$, there is an active attack (S''', d) in F with $S''' \cup \{d\} = S'' \cup \{c\}$. But now again (S, b) is inactive in $F \cup H$, a contradiction. Otherwise if (S', b) is active in G then, by $F^{sk} = G^{sk}$, there is an active attack (S'', c) in F with $S'' \cup \{c\} = S'' \cup \{b\}$. Hence (S, b) is inactive in F , a contradiction.

- If (S, b) is inactive, i.e. there is $S' \subset S, c \in S$ with $(S', c) \in R(G \cup H)$ and (S', c) active. As (S, b) is active in $F \cup H$ we have that $(S', a) \in R(G)$. By $F^{sk} = G^{sk}$, there is an active attack (S'', d) in F with $S'' \cup \{d\} = S' \cup \{c\}$. Thus (S, b) is inactive in F , a contradiction.

We obtain that (S, b) is active and non-redundant and thus $(S, a) \in R((G \cup H)^{sk})$. \square \square

While the previous lemmas enable us to show that two SETAFs with the same kernel are strongly equivalent it remains to show that this condition is necessary. We do so in the next theorem by providing constructions for a (SET)AF H that shows that two SETAFs are not strongly equivalent if they have different kernels. Moreover, we extend our results to the other notions of equivalence.

Theorem 1. *For any AFs F and G and $\sigma \in \{stb, stage\}$ the following are equivalent: (a) $F \equiv_s^\sigma G$; (b) $F \equiv_n^\sigma G$; (c) $F \equiv_{sd}^\sigma G$; (d) $F \equiv_{nd}^\sigma G$; (e) $F^{sk} = G^{sk}$.*

Proof. By definition (a) implies (b) and (c). Likewise, (b) implies (d) and (c) implies (d). It remains to show (1) $F^{sk} = G^{sk}$ implies $F \equiv_s^\sigma G$ and (2) $F^{sk} \neq G^{sk}$ implies $F \not\equiv_{nd}^\sigma G$. By Lemma 1 we can assume that F and G are in minimal form.

(1) Suppose $F^{sk} = G^{sk}$ and let H, S such that $S \in \sigma(F \cup H)$. We show $S \in \sigma(G \cup H)$. By Lemma 3, $S \in \sigma((F \cup H)^{sk})$ and we get $S \in \sigma((G \cup H)^{sk})$ from Lemma 4. Thus, $S \in \sigma(G \cup H)$, again by Lemma 3. By symmetry and definition of strong equivalence, we get $F \equiv_s^\sigma G$.

(2) First, we consider the case $A(F^{sk}) \neq A(G^{sk})$. This implies $A(F) \neq A(G)$. W.l.o.g. let $a \in A(F) \setminus A(G)$. We use $B = (A(F) \cup A(G)) \setminus \{a\}$, and c as a fresh argument. Consider $H = (B \cup \{c\}, \{(c, b) \mid b \in B\})$. Note that H is conform with the definition of \equiv_{nd}^σ , i.e. it is a simple AF not changing the relation between existing arguments. Suppose now, a is contained in some $S \in \sigma(F \cup H)$. Then, we are done since a cannot be contained in any $S' \in \sigma(G \cup H)$, since $a \notin A(G \cup H)$. Otherwise, we extend H to $H' = H \cup (\{a\}, \emptyset)$. Then, $\{a, c\}$ is the unique stable extension (and thus unique stage extension) of $G \cup H'$. On the other hand, observe that $F \cup H' = F \cup H$, hence by assumption, a is not contained in any $S \in \sigma(F \cup H')$. In both cases, we get $F \not\equiv_s^\sigma G$. Now suppose $A(F^{sk}) = A(G^{sk})$ but $R(F^{sk}) \neq R(G^{sk})$. W.l.o.g. assume there exists some $(S, a) \in R(F^{sk}) \setminus R(G^{sk})$ such that there is no $(S', a) \in R(G^{sk})$ with $S' \subset S$ (otherwise exchange the roles of F and G). We distinguish the two cases of attacks that constitute the stable kernel: (1) $(S, a) \in R$ is active in F with $a \notin S$; (2) (S, a) with $a \in S$, such that there is some $(S, d) \in R$ with $d \in S$ active in F .

1) For fresh arguments c, t , we define $H = (A(F) \cup \{c, t\}, R_H)$ with

$$R_H = \{(t, c), (c, t)\} \cup \{(c, b) \mid b \in A(F) \setminus (S \cup \{a\})\} \cup \{(t, b) \mid b \in A(F)\}.$$

First, by construction we have that $\{t\} \in stb(F \cup H)$ and $\{t\} \in stb(G \cup H)$ and thus stable and stage semantics coincide in both $F \cup H$ and $G \cup H$. Thus we can restrict ourselves to stable semantics. We have $S \cup \{c\} \in stb(F \cup H)$, since $S \cup \{c\}$ is conflict-free and attacks all arguments $b \notin S$ either collectively via S or via the newly introduced argument c . However, $S \cup \{c\} \notin stb(G \cup H)$ as by the assumption there is no $(S', a) \in R(G^{sk})$ with $S' \subseteq S$ and thus $S \cup \{c\}$ does not attack a .



Figure 3: Given that $(\{a, b\}, c)$ is inactive it is equivalent to $(\{a, b, c\}, c)$ for all semantics under our considerations. We call the attack $(\{a, b, c\}, c)$ in F reducible and G the normal form of F .

2) Notice that, by construction, whenever $(S, a) \in R(F^{sk})$ then also $(S, b) \in R(F^{sk})$ for all $b \in S$. W.l.o.g we can assume that there is no $(S', b) \in R(G^{sk})$ with $S' \cup \{b\} \subseteq S$ (otherwise we exchange the roles of F and G as $(S', b) \notin R(F^{sk})$). For a fresh argument c , we define

$$H = (A(F) \cup \{c\}, \{(c, b) \mid b \in A(F) \setminus S\}).$$

We have $S \cup \{c\} \notin stb(F \cup H)$ and $S \cup \{c\} \notin stage(F \cup H)$, since (S, a) is a conflict within the set $S \cup \{c\}$. However, for $G \cup H$ we have that $S \cup \{c\}$ is conflict free and attacks all argument outside the set, i.e. $S \cup \{c\} \in stb(G \cup H)$ and thus also $S \cup \{c\} \in stage(G \cup H)$.

In both cases we have found a witness H for $F \not\equiv_{nd}^\sigma G$. \square \square

4.2 SETAFs in Normal Form

We next turn to admissible based semantics, i.e. *adm*, *com*, *pref*, and *sem* semantics, and define the respective kernels. While for stable semantics we can ignore inactive attacks they are significant for admissible-based semantics as one has to defend arguments also against inactive attacks. We first identify equivalent inactive attacks and introduce a corresponding normal form of SETAFs.

Definition 10. Let $F = (A, R)$ be a SETAF. We call an attack (S, a) with $a \in S$ reducible in F if there exists $S' \subseteq S \setminus \{a\}$ and $b \in S$ such that $(S', b) \in R$.

First note that a reducible attack (S, a) is inactive since the set S is conflicting; thus S will never appear in a conflict-free set T . Moreover, each conflict-free set T which defends the argument a attacks some argument in $S \setminus \{a\}$, otherwise T would be conflicting. We introduce a normal form of a SETAF F which is given by its minimal form where each reducible attack (S, a) is replaced by the attack $(S \setminus \{a\}, a)$. Figure 3 shows a SETAF F and its normal form G . Here, $(\{a, b, c\}, c)$ is reducible in F ; the attack is replaced by $(\{a, b\}, c)$ in G .

Definition 11. Let $F = (A, R)$ be a SETAF. We define the normal form G of F as the minimal form of $(A, R \cup \{(S \setminus \{a\}, a) \mid (S, a) \text{ reducible in } F\})$.

The next lemma states that replacing reducible attacks (S, a) with $(S \setminus \{a\}, a)$ preserves the semantics. The modification does not affect conflict-free sets; furthermore, the argument a is defended by the same conflict-free sets in both SETAFs F and its normal form G . Moreover, modifying inactive attacks does not affect stable and stage extensions. This follows directly from Lemma 3 and the fact that inactive attacks are deleted in the stable kernel.

Lemma 5. Let $F = (A, R \cup \{(S, a)\})$ and let (S, a) be reducible in F . Let $G = (A, R \cup \{(S \setminus \{a\}, a)\})$. Then $\sigma(F) = \sigma(G)$ for $\sigma \in \{\text{adm}, \text{pref}, \text{sem}, \text{com}, \text{stb}, \text{stage}\}$.

It follows that each SETAF F and its normal form G are also strongly equivalent. Indeed, consider an extension $F \cup H$ where H is arbitrary. The repetitive application of Lemma 5 yields $\sigma(F \cup H) = \sigma(G \cup H)$ for all considered semantics σ .

Proposition 1. For a SETAF F and its normal form G , $F \equiv_s^\sigma G$ for $\sigma \in \{\text{adm}, \text{pref}, \text{sem}, \text{com}, \text{stb}, \text{stage}\}$.

4.3 Admissible & Complete Kernel

We start this section by introducing the kernel for complete semantics. The complete kernel F^{ck} consists of all active attacks and inactive attacks (S, a) such that a is not attacked by any $S' \subset S \cup \{a\}$. Notice that whenever there is such an attack (S', a) the argument a is only defended by a complete extension E if E attacks $S' \setminus \{a\}$ and thus also S , i.e. whenever a is defended against (S', a) it is also defended against (S, a) . We thus do not include such attacks (S, a) in the kernel. It turns out that all the remaining inactive attacks influence whether the argument a is defended by a set E or not for certain expansions H .

Definition 12. For a SETAF $F = (A, R)$ in normal form, we define the complete kernel of F as $F^{ck} = (A, R^{ck})$ with

$$R^{ck} = \{(S, a) \in R \mid (S, a) \text{ is active in } F\} \cup \{(S, a) \in R \mid \nexists S' \subset S \cup \{a\} : a \in S', (S', a) \in R\}.$$

The complete kernel of an arbitrary SETAF F is the complete kernel of the normal form of F .

For admissible semantics, we extend the complete kernel by additionally removing inactive attacks (S, a) where the attacked argument a defends itself against S . Notice that self-defense is not sufficient for removing an inactive attack in the complete kernel since inactive attacks must be additionally taken into account for determining whether arguments outside of an admissible set T are defended by this set T .

Definition 13. For a SETAF $F = (A, R)$ in normal form, we define the admissible kernel of F as $F^{ak} = (A, R^{ak})$ with

$$R^{ak} = \{(S, a) \in R \mid (S, a) \text{ is active in } F\} \cup \{(S, a) \in R \mid \nexists S' \subset S \cup \{a\} : a \in S' \text{ and } (S', a) \in R\} \cap \{(S, a) \in R \mid \nexists b \in S \text{ such that } (\{a\}, b) \in R\}.$$

The admissible kernel of an arbitrary SETAF F is the admissible kernel of the normal form of F .

Example 1. Consider the SETAF $F = (A, R)$ from Figure 4, which shows F together with its complete and its admissible kernel. Attacks which are colored in red are inactive. The complete kernel F^{ck} is constructed by removing the inactive attack $(\{d, e\}, c)$ since c is attacked by $\{c, d\}$, i.e. by a subset of $\{c, d, e\}$. In the admissible kernel F^{ak} also the attack $(\{a, b\}, c)$ can be removed, since c defends itself by attacking the argument a . Observe that the set $\{e\}$ is admissible and complete in both F and F^{ck} but $\{e\}$ is not complete in F^{ak} since e defends c in F^{ak} .

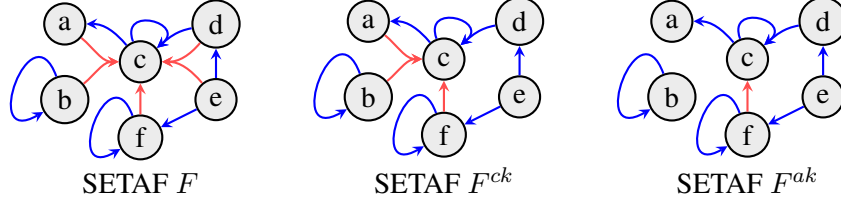


Figure 4: Complete and the admissible kernel of a SETAF F . Active attacks in blue. Inactive in red.

Before showing our characterisation for strong equivalence we clarify the relation between the introduced kernels. Observe that $F^{ak} \subseteq F^{ck}$ by definition.

Lemma 6. *For any two SETAFs F, G in normal form, (a) $F^{ak} = G^{ak}$ implies $F^{sk} = G^{sk}$ and (b) $F^{ck} = G^{ck}$ implies $F^{ak} = G^{ak}$ and $F^{sk} = G^{sk}$.*

Proof. We will show (a) and omit the proof of (b) due to space limits. Assume that $F^{ak} = G^{ak}$. We show that $F^{sk} \subseteq G^{sk}$, the other direction is by symmetry. Let $(S, a) \in R(F^{sk})$. We show that $(S, a) \in R(G^{sk})$.

Towards a contradiction, assume $(S, a) \notin R(G^{sk})$. First note that we can assume that (S, a) is active in F . In the case $(S, a) \in \{(S, b) \mid \exists(S, c) \text{ active in } F, b, c \in S\}$, there is an active attack $(S, b) \in R(F)$, $b \in S$, and $(S, b) \notin R(G^{sk})$ (otherwise there is an active attack $(S, c) \in R(G)$, $c \in S$, and therefore $(S, a) \in R(G^{sk})$, contradiction).

By definition of F^{ak} , we get that $(S, a) \in F^{ak}$, and therefore $(S, a) \in R(G^{ak})$ by assumption. Thus $(S, a) \in R(G)$ and (S, a) is inactive (since $(S, a) \notin R(G^{sk})$). By Lemma 2, there is an active attack $(S', b) \in R(G)$ such that $S' \subseteq S$, $b \in S$. Thus we conclude that $(S', b) \in R(G^{ak})$ (by definition of the admissible kernel) $(S', b) \in R(F^{ak})$ (by assumption $F^{ak} = G^{ak}$) and therefore $(S', c) \in R(F)$, making the attack (S, a) inactive in F , contradiction. $\square \quad \square$

Two SETAFs are strongly equivalent w.r.t. *com* semantics iff their complete kernels coincide. Likewise two SETAFs are strongly equivalent w.r.t. *adm*, *pref*, or *sem* semantics iff their admissible kernels coincide. The proofs proceed in a similar way as for stable kernels.

Lemma 7. *For any SETAF $F = (A, R)$, $com(F) = com(F^{ck})$, and $\sigma(F) = \sigma(F^{ak})$ for $\sigma \in \{adm, pref, sem\}$.*

The next lemma states that if two SETAFs have the same kernel then their extensions with an arbitrary SETAF H will also agree on their kernels.

Lemma 8. *Let F, G be SETAFs in normal form. For all SETAFs H , (a) if $F^{ck} = G^{ck}$ then $(F \cup H)^{ck} = (G \cup H)^{ck}$ and (b) if $F^{ak} = G^{ak}$ then $(F \cup H)^{ak} = (G \cup H)^{ak}$.*

Using the previous lemmas one can show that two SETAFs F, G are strongly equivalent w.r.t. complete semantics iff their complete kernels coincide. It can be shown that the conditions are also necessary and characterize other notions of equivalence as well.

Theorem 2. *For any two SETAFs F, G , the following are equivalent: (a) $F \equiv_s^{com} G$; (b) $F \equiv_n^{com} G$; (c) $F \equiv_{sd}^{com} G$; (d) $F \equiv_{nd}^{com} G$; (e) $F^{ck} = G^{ck}$.*

Similarly, any two SETAFs F, G with the same admissible kernel are strongly equivalent w.r.t. admissible, preferred and semi-stable semantics.

Theorem 3. *For any two SETAFs F, G and for $\sigma \in \{adm, pref, sem\}$, the following are equivalent: (a) $F \equiv_s^\sigma G$; (b) $F \equiv_n^\sigma G$; (c) $F \equiv_{sd}^\sigma G$; (d) $F \equiv_{nd}^\sigma G$; (e) $F^{ak} = G^{ak}$.*

Due to space limits, we shall omit the proofs of the above theorems. Instead, we highlight central constructions and sketch the main arguments. The proofs proceed in the same way as the proof of Theorem 1, i.e. first we use the Lemmas 7 & 8 to show $(e) \Rightarrow (a)$. To show $\neg(e) \Rightarrow \neg(d)$, we assume that the kernels of F and G differ and then construct an AF H such that $\sigma(F \cup H) \neq \sigma(G \cup H)$. This again requires a case study where the crucial new arguments are for the cases where the argument sets of the kernels coincide but there is an inactive attack (S, a) which is just present in the kernel of F but not in the kernel of G (or vice versa). W.l.o.g. we can assume that (S, a) is a minimal such attack. We sketch the case where $a \notin S$ below. There we use the following AF

$$H = (A(F) \cup \{c, d\}, \{(c, b) \mid b \in A(F) \setminus (S \cup \{a\})\} \cup \{(d, d), (d, b) \mid b \in S\}).$$

Note that, as (S, a) is inactive, S contains an attack (S', b) for some $S' \subseteq S, b \in S$. For admissible kernels the set $\{a, c\}$ is not admissible in $F \cup H$ as the argument a is not defended against S . On the other hand, it can be shown that $\{a, c\}$ is admissible in $G \cup H$: Clearly, $\{a, c\}$ is conflict-free; moreover, the argument c defends a in $G \cup H$ against every attack where the source-set contains arguments from $A(F) \setminus (S \cup \{a\})$. It can be shown that there is no attack (T, a) with $T \subseteq S \cup \{a\}$ using the definition of admissible kernels and the fact that (S, a) is minimal among the attacks in the symmetric difference of the kernels F and G . For complete kernels, $\{c\}$ is complete in $F \cup H$ as a is not defended against S . On the other hand, we have that $\{c\}$ is not complete in $G \cup H$ since one can show that c defends a using the definition of complete kernels and the fact that the kernels of F and G coincide on smaller attacks. That is, for both kernels the AF H is a witness of $F \not\equiv_{nd}^\sigma G$ for the corresponding semantics σ .

5 Conclusion

In this work we considered strong equivalence for SETAFs under admissible, complete, preferred, stable, semi-stable and stage semantics. Strong equivalence between SETAFs can be characterized by computing so-called kernels and comparing them on a syntactical level. By that, strong equivalence for the considered semantics can be tested in polynomial time. Moreover, the SETAF kernels are generalizations of the respective kernels in the AF setting, in the sense that when applied to AFs our kernels coincide with the ones from [13]. Given the relations between kernels for SETAFs F, G we obtain that the strong equivalence notions of the different semantics coincide as follows: $F \equiv_s^{stb} G \Leftrightarrow F \equiv_s^{stage}$; and $F \equiv_s^{adm} G \Leftrightarrow F \equiv_s^{pref} G \Leftrightarrow F \equiv_s^{sem} G$. Moreover, (a) whenever $F \equiv_s^{com} G$ then also $F \equiv_s^\sigma G$ for all $\sigma \in \{adm, pref, stb, sem, stage\}$, and (b) whenever $F \equiv_s^\tau G$ for $\tau \in \{adm, pref, sem\}$ then also $F \equiv_s^\sigma G$ for all $\sigma \in \{stb, stage\}$.

One finding based on the kernels is that strongly equivalent SETAFs necessarily coincide w.r.t. their set of arguments, which is in accordance with the results for Dung AFs. However, we identified classes of attacks that can be removed without affecting strong equivalence. Notice that this goes beyond the notion of redundant attacks in SETAFs from [14]. In particular a significant difference between the SETAF setting and the AF setting appears when we consider frameworks without self-attacks. For AFs without self-attacks the kernels coincide with the initial AFs while for SETAFs the kernels, even in absence of self-attack, remove (certain) inactive attacks. The reason for this is that the only way to deactivate an attack in AFs is to make the source argument self-attacking while in SETAFs there several ways to produce a conflict in the source set of an attack.

One direction for future work is to extend our results to further semantics as ideal, eager and grounded semantics. Notice that although grounded semantics is closely related to admissible and complete semantics neither the admissible nor the complete kernel are suitable to characterize strong equivalence w.r.t. grounded semantics. This is immediate by the corresponding results of AFs where the grounded kernel is different from all the other kernels [13]. Another direction for future research are generalizations of alternative notions of equivalence that have been investigated for AFs, e.g. the recently introduced notion of C -relativized equivalence [4].

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