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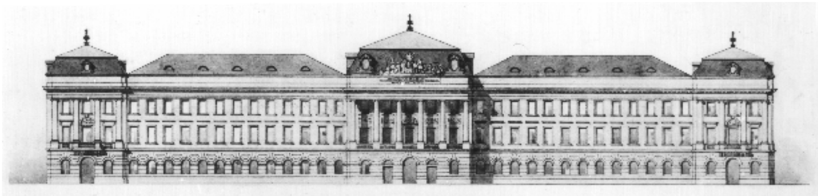

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Splitting Argumentation Frameworks with Collective Attacks

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Splitting Argumentation Frameworks with Collective Attacks

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Abstract. A recurring notion in the abstract argumentation community is that of collective attacks, whereby a set of arguments, rather than a single one, attacks another argument. The resulting frameworks capturing this phenomenon are referred to as SETAFs. Given the possibility of facing an exponential runtime in the size of the framework, techniques have been presented to compute extensions of a given framework incrementally, i.e. restricting the search space to sub-frameworks only, and then combining the obtained results. Existing research has primarily focused on approaches based on SCC-recursiveness, where SETAFs are evaluated along their strongly connected components (SCCs) using generalized semantics and dedicated algorithms. Splitting approaches are more general in this regard, as they do not have to consider SCCs individually and can be used on top of arbitrary argumentation solvers. Splitting techniques have been successfully applied in abstract argumentation but have been neglected for SETAFs so far. Towards filling this gap our work investigates the concept of (modification-based) splitting for SETAFs. We show that a splitting-based approach is possible for common semantics (such as admissible, complete, grounded, preferred, and stable), generalizing corresponding results of AFs, which can be seen as a restricted class of SETAFs. Along the way, we point out intricate details that are obvious or trivial for AFs, but help us to understand the underlying ideas in greater detail than before.

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1 Introduction

Most argumentation problems are intractable in general, which means even the best-known methods for solving these problems oftentimes cannot avoid exploring the whole exponential size solution-space. The fewer arguments we have to consider for each problem, the more efficient an approach can be in general, which is why in the setting of abstract argumentation (see [9]) it can be advantageous to evaluate only *parts* of a framework at once and then combine the results. One approach to do this is *splitting*, as introduced by Baumann for Dung-style argumentation frameworks (AFs) [2] and later considered in other settings [16, 3, 4]. A popular syntactic addition within argumentation frameworks that has been regarded in the community are collective attacks (due to Nielsen and Parsons [17]), the resulting frameworks are called SETAFs [13, 12, 5]. Collective attacks have proven useful for the instantiation of structured argumentation, see e.g. [7, 14, 8]. However, a splitting approach for SETAFs has not yet been introduced; in this paper we close this gap and investigate interesting differences and similarities to the simpler AF-case, which ultimately shines a new light on the existing ideas on splitting. We show that if we carefully generalize the underlying intuitions, SETAFs yield an elegant splitting procedure for most common semantics. Splitting techniques have also been introduced for Abstract Dialectical Frameworks (ADFs) [16]. However, even though SETAFs can be modeled through ADFs, it is not obvious how these results are applicable in the context of SETAFs—as ADFs deal with simple links and propositional acceptance conditions whereas SETAFs rely on directed hypergraphs.

Similar to the structure of [2], on this work, we consider *directional splitting*, which means that we split a large SETAF SF into two sub-frameworks SF_1 and SF_2 in a way s.t. only SF_1 influences SF_2, but not the other way around. Clearly, the choices within the subframework SF_1 determine the acceptable arguments in SF_2, which has to be accounted for in order to correctly solve reasoning problems. In general, for such a scenario there are two approaches: (a) SF_2 is evaluated w.r.t. generalized semantics, that take the decisions in SF_1 into account, or (b) the modification-based approach, i.e., SF_2 is syntactically modified, to simulate the effects of the decisions in SF_1. While (a) is used for example in SCC-recursiveness [12] (i.e., an approach based on the strongly-connected components), AFs with input/output [1], or decomposition-based techniques for AFs [15], approach (b) allows us to use unmodified out-of-the-box argumentation solvers on both sub-frameworks. In this paper we will present a modification-based splitting approach for SETAFs. This paper is organized as follows.

- In Section 2 we recall the definition of SETAFs and their semantics, as well as the splitting approach in AFs.
- Throughout Section 3 we explain the intuitions and intricacies of splitting in the context of collective attacks and justify our design choices for the splitting algorithm.
- Section 4 contains the resulting definition of the splitting approach for SETAFs, as well as the theoretical underpinning for its correctness. Moreover, we establish its connection to the *directionality* principle—a desirable result we already know from AFs.
- Finally, in Section 5 we conclude and discuss related work.

2 Background

In this section, we recall the definition of argumentation frameworks with collective attacks (SETAFs) [17, 13, 5] and their semantics. As we will see, with a slight abuse of notation, we can view Dung-style argumentation frameworks (AFs) [9] as a special case of SETAFs.

Definition 1. *A SETAF is a pair $SF = (A, R)$ where A is a finite set of* arguments*, and $R \subseteq (2^A \setminus \{\emptyset\}) \times A$ is the* attack relation*. For an attack $(T, h) \in R$ we call T the* tail *and h the* head *of the attack. SETAFs (A, R), where for all $(T, h) \in R$ it holds that $|T| = 1$, amount to (standard Dung) AFs. We usually write (t, h) to denote the set-attack $(\{t\}, h)$. For $S \subseteq A$, we say S* attacks *an argument $a \in A$ if there is an attack $(T, a) \in R$ with $T \subseteq S$. Moreover, for a set $B \subseteq A$ we say that S attacks B if S attacks some $b \in B$. We use S^+_R to denote the set $\{a \mid S \text{ attacks } a\}$ and define the* range *of S (w.r.t. R), denoted $S^\oplus_R$, as the set $S \cup S^+_R$.*

The fundamental notions of conflict and defense from Dung-style AFs naturally generalize to SETAFs. These notions are the basis for the semantics we investigate in this paper.

Definition 2. *Let $SF = (A, R)$ be a SETAF. A set $S \subseteq A$ is* conflicting *in SF if S attacks a for some $a \in S$. $S \subseteq A$ is* conflict-free *in SF, if S is not conflicting in SF, i.e. if $T \cup \{h\} \not\subseteq S$ for each $(T, h) \in R$. $cf(SF)$ denotes the set of all conflict-free sets in SF. An argument $a \in A$ is* defended *(in SF) by a set $S \subseteq A$ if for each $B \subseteq A$, such that B attacks a, also S attacks B in SF. A set $T \subseteq A$ is defended (in SF) by S if each $a \in T$ is defended by S (in SF).*

The semantics we study in this work are the admissible, complete, grounded, preferred, and stable semantics, which we will abbreviate by *adm*, *com*, *grd*, *pref*, and *stb*, respectively [17, 13, 10]. Moreover, we use $\sigma(SF)$ to denote the set of *extensions* of SF under semantics σ.

Definition 3. *Given a SETAF $SF = (A, R)$ and a conflict-free set $S \in cf(SF)$. Then,*

- *$S \in adm(SF)$, if S defends itself in SF,*
- *$S \in com(SF)$, if $S \in adm(SF)$ and $a \in S$ for all $a \in A$ defended by S,*
- *$S \in grd(SF)$, if $S \in com(SF)$ and there is no $T \in com(SF)$ s.t. $T \subset S$,*
- *$S \in pref(SF)$, if $S \in adm(SF)$ and there is no $T \in adm(SF)$ s.t. $T \supset S$, and*
- *$S \in stb(SF)$, if $S^\oplus_R = A$.*

The relationship between the semantics has been clarified in [17, 13, 10] and matches with the relations between the semantics for Dung AFs, i.e. for any SETAF SF:

$$stb(SF) \subseteq pref(SF) \subseteq com(SF) \subseteq adm(SF) \subseteq cf(SF)$$

We now recall Baumann's splitting approach for AFs [2] (in a slightly adapted equivalent form to fit our notation).

Definition 4. *Let $F = (A, R)$ be an AF, $F_1 = (A_1, R_1)$ and $F_2 = (A_2, R_2)$ two sub-frameworks of SF s.t. $A_1 \cap A_2 = \emptyset$, $A = A_1 \cup A_2$ and $R = R_1 \cup R_2 \cup R_3$ with $R_3 \subseteq A_1 \times A_2$. We call the triple (F_1, F_2, R_3) a* splitting *of F. For such a splitting the (E, R_3)*-reduct *w.r.t. $E \subseteq A_1$ is the AF $AF' = (A', R')$ with $A' = A_2 \setminus E^+_{R_3}$ and $R' = R_2 \cap (A' \times A')$. The set of* undecided arguments *w.r.t. $E \subseteq A_1$ is defined as $U_E = A_1 \setminus E^{\oplus}_{R_1}$.*

We will later generalize the notion of the reduct to be applicable in the context of SETAFs.

Definition 5. *Let (F_1, F_2, R_3) be a splitting for an AF F and E an extension of F_1. Moreover, take $F'_2 = (A'_2, R'_2)$ as the (E, R_3)-reduct of F_2 and U_E as the set of undecided arguments w.r.t. E. The (U_E, R_3)-modification of F_2 is defined as $mod_{U_E,R_3}(F'_2) = (A'_2, R'_2 \cup \{(b, b) \mid \exists a \in U_E : (a, b) \in R_3\})$.*

It is easy to see that the definition of the modification does not actually rely on the undecided *arguments*, but rather uses the arguments as means to obtain the *links* which stem from undecided arguments. Later on, we will make use of this fact to simplify the respective notions. Baumann [2] showed that by this definition it is possible for a splitting to compute the extensions for each sub-framework separately.

Theorem 6 ([2]). *Let (F_1, F_2, R_3) be a splitting for an AF $F = (A, R)$ with $F_i = (A_i, R_i)$ and $\sigma \in \{cf, adm, stb, com, pref, grd\}$.*

1. *If $E_1 \in \sigma(F_1)$ and $E_2 \in \sigma(mod_{U_E,R_3}(F'_2))$, then $E_1 \cup E_2 \in \sigma(F)$.*
2. *If $E \in \sigma(F)$, then $E \cap A_1 \in \sigma(F_1)$ and $E \cap A_2 \in \sigma(mod_{U_E,R_3}(F'_2))$.*

3 Towards Splitting for SETAFs

In this section, we introduce fundamental ideas for defining divide and conquer algorithms based on splitting in the presence of collective attacks. As a starting point, we generalize the notion of splitting introduced in [2] for Dung-style AFs. For this, we will in the following provide comprehensive intuitions.

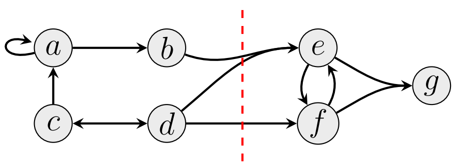
Definition 7. *Let $SF = (A, R)$ be a SETAF, $SF_1 = (A_1, R_1)$ and $SF_2 = (A_2, R_2)$ two sub-frameworks of SF such that $A_1 \cap A_2 = \emptyset$, $A = A_1 \cup A_2$ and $R = R_1 \cup R_2 \cup R_3$ with $R_3 \subseteq \big((2^{A_1} \setminus \{\emptyset\}) \cup 2^{A_2}\big) \times A_2$. We call a* splitting *of SF the triple (SF_1, SF_2, R_3). Moreover, we say that R_3 is the set of* links *of the splitting (SF_1, SF_2, R_3).*

In a nutshell, we investigate a large SETAF SF that has two sub-frameworks SF_1 and SF_2 with attacks within themselves, and attacks R_3 that stem from A_1 (at least in part) and target only arguments in SF_2. The general idea is to compute extensions of SF_1 and SF_2, which combined give us extensions of SF. Due to the links from SF_1 to SF_2 we have to modify SF_2 according to the extension(s) of SF_1 to account for the prior accepted and rejected arguments.

3.1 Simple Splitting for SETAFs

We take here into account SETAF splittings where the *whole* tail of the links is separated from the respective heads. Notice that, whereas these are indeed straightforward generalizations of AF splittings [2], they represent only a specific class of the splittings introduced by Definition 7. In Section 3.2, we will investigate the problem of splitting SETAFs in full generality.

Example 8. *Consider the SETAF SF below with its splitting (SF_1, SF_2, R_3), with $SF_1 = (A_1, R_1)$ and $SF_2 = (A_2, R_2)$. The dashed line indicates the separation between the sub-frameworks. The goal is to compute the extensions of SF by computing the extensions of SF_1 and (a modified version of) SF_2 separately. Note that throughout the remaining part of this paper we will refer to the splitting in question always as (SF_1, SF_2, R_3), unless indicated otherwise.*



$A_1 = \{a, b, c, d\}$
$R_1 = \{(a, a), (a, b), (c, d), (d, c), (c, a)\}$
$A_2 = \{e, f, g\}$
$R_2 = \{(e, f), (f, e), (\{e, f\}, g)\}$
$R_3 = \{(\{b, d\}, e), (d, f)\}$

One can verify that SF_1 has two preferred extensions: $E_1 = \{b, c\}$ and $E_2 = \{d\}$. Let us first consider E_1. Since d is defeated in SF_1 we can argue that the links (i.e., attacks in R_3) do not at all affect the arguments in SF_2, and we can just evaluate SF_2 "as is" to obtain the extensions $E_{1,1} = \{e, g\}$ and $E_{1,2} = \{f, g\}$. We can combine the extensions from SF_1 and SF_2 to obtain $\{b, c, e, g\}$ and $\{b, c, f, g\}$, which are indeed preferred extensions of SF. Now on the other hand if we consider E_2, it is non-trivial how this affects SF_2: while f is defeated by d, e is targeted by an attack from the accepted argument d and the argument b, which in SF_1 is neither accepted nor outright rejected by E_2. In the following, we will argue how to properly deal with these cases and introduce a splitting method that correctly characterizes all extensions.

In the example above we can see that the status of the arguments in SF_1 w.r.t. the extension of SF_1 we investigate determines whether and how the arguments in SF_2 are affected. Indeed, in the AF case the status of the single argument in the tail of a link solely determines whether the head is removed or not from the second sub-framework, or whether a self-attack is added to it (c.f. Definitions 4 and 5). Similarly for SETAFs, it is possible to distinguish three different scenarios for the status of the arguments in A_1 after evaluating SF_1, corresponding to the cases where the argument is accepted (i.e., in an extension $E_1 \in \sigma(SF_1)$), defeated (in $(E_1)^+_{R_1}$) or undecided (in $A_1 \setminus (E_1)^{\oplus}_{R_1}$). Note however, that while on AFs the status of a link and its one tail argument coincide, for SETAFs links (like any other attack) can have *multiple* tail arguments. Hence, the status of a link $(T, h) \in R_3$ of a splitting (SF_1, SF_2, R_3) can be determined after evaluating SF_1 as follows: (i) *all* of the arguments in the tail of a link (that are also in SF_1) are accepted (i.e., in an extension $E_1 \in \sigma(SF_1)$), (ii) at least one argument in the tail of a link is defeated by E_1 (in $(E_1)^+_{R_1 \cup R_3}$) or (iii) no argument is defeated but at least one is undecided (in $A_1 \setminus E^{\oplus}_{R_1}$). In what follows, we consider cases (i)-(iii) separately and show an intuition on how these need to be treated. We start with (i) in the Example 9, depicting a situation where the attack $(\{x, y\}, z)$ is *in*. As one can see, in certain circumstances the above definition of a splitting corresponds to a straightforward generalization of the AF case.

Example 9. *Consider the AF F (left), and the SETAF SF (right) with its splitting (SF_1, SF_2, R_3).*

We look at the preferred extensions $\{a\}$ for the first part of F, and see that b in the second part is defeated. Thus, by the approach of [2], b is removed when we look at the modified second framework, and its outgoing attack towards c as well. We trivially obtain $\{c\}$ as a preferred extension for the modified right part. We combine $\{a\} \cup \{c\}$ to obtain $\{a, c\}$ as the only preferred extension for F.

Analogously, we have $\{v, w\}$ as a preferred extension in SF_1. Hence, we remove x from SF_2 to obtain the modified framework $SF_2^\star$. As in the AF case, any outgoing attack from x (no matter if other arguments are in the tail, like y in our case) cannot affect their targeted argument (z), as z is defended against this attack. Hence, we remove the entire attack and obtain $SF_2^\star = (\{y, z\}, \emptyset)$ which trivially yields $\{y, z\}$ as its preferred extension. As a result, one gets $\{v, w\} \cup \{y, z\}$ as a preferred extension for SF.

We next discuss case (ii) for the status of a link in Example 10 below.

Example 10. *For the SETAF SF below, we identify the preferred extension $\{w, y\}$ in the first part (i.e., $\{w, y\} \in$ pref(SF_1)). Given that $w \in E$, we get that x is defeated, which means the link $(\{x, y\}, z)$ can be seen as* out. *Intuitively, z needs no more counter-attack for this incoming attack—since x is already defeated. For this reason, such an attack does not affect the modification of SF_2. Hence, the modified $SF_2^\star$ is as depicted on the right, preserving z as an acceptable argument. Note also that any possible outgoing attacks of z remain untouched. We thus have $\{w, y\} \in$ pref(SF_1) and $\{z\} \in$ pref$(SF_2^\star)$ as preferred extensions of SF_1 and $SF_2^\star$, respectively. It is easy to see that $\{w, y\} \cup \{z\}$ is a preferred extension of SF.*

Different considerations are due whenever the original SETAF contains a link which is *undecided*, corresponding to case (iii).

Example 11. *Consider the SETAF SF displayed below (left) and $E = \{y\} \in$ pref(SF_1). x is undecided w.r.t. E, i.e., $x \in A_1 \setminus E^{\oplus}_{R_1}$. This makes the status of $(\{x, y\}, z)$ undecided as well, enforcing a modification of the right part of the SETAF. Analogous to the approach for AFs, we add a self-attack on z, obtaining $SF_2^\star = (\{z\}, \{(z, z)\})$ (right). Intuitively, this models the fact that z cannot be accepted in $SF_2^\star$, since there is at least one attack that z is not defended against.*

On the other hand, z is not rejected, i.e., if z were to attack other arguments in SF_2 they need defence against z. Hence, we cannot outright remove z as in case (ii). In this situation, one gets $\emptyset \in pref(SF_2^{\star})$. Hence, $\{y\} \cup \emptyset$ is a preferred extension of SF.

Examples 9, 10, and 11 above display only a special case of splitting for SETAFs, where the *whole* tail of an attack is separated from its target. Hence, as of now performing splitting-based techniques in the presence of collective attacks represents an easy and straightforward generalization of its AF counterpart. This stems from the fact that modification is only slightly impacted by the presence of multiple arguments in the tail of a link. Consequently, minor adjustments are needed to handle such situations. However, due to their rich syntax, SETAFs allow for another possible way to separate an attack via splitting. In the following, we consider SETAF splittings in full generality.

3.2 Diagonal Splitting for SETAFs

The enriched syntax of SETAFs allows us to take into account splittings that separate arguments taking part in the same collective attack. In particular, it is possible to split a SETAF in such a way that two parts of the same tail of a link end up being in different sub-frameworks. This is captured by the possibility of having $R_3 \subseteq \big((2^{A_1} \setminus \{\emptyset\}) \cup 2^{A_2}\big) \times A_2$ as for our Definition 7. We investigate such scenarios in connection with the cases (i)-(iii) as before. Note however that we need to consider the cases more carefully, as we now also consider links $(T, h) \in R_3$ where the tail T is spread over both A_1 and A_2. More formally we call a link $(T, h) \in R_3$:

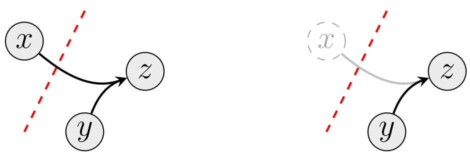
(i) *in* iff $\nexists a \in T$ s.t. $a \in E^+_{R_1 \cup R_3}$ and $\forall a \in T \cap A_1, a \in E$,

(ii) *out* iff $\exists a \in T$ s.t. $a \in E^+_{R_1 \cup R_3}$ or

(iii) *undec* iff $\nexists a \in T$ s.t. $a \in E^+_{R_1 \cup R_3}$ and $\exists a \in T \cap A_1$ s.t. $a \in A \setminus E^{\oplus}_{R_1}$.

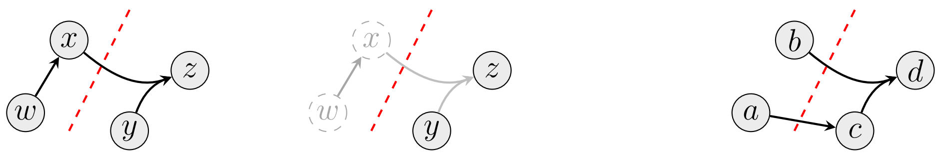
We again give the necessary intuitions for each case, starting with (i).

Example 12. *For the below SETAF SF (left), we consider $\{x\} \in adm(SF_1)$. Therefore, intuitively it is the remaining part of the attack stemming from y that is decisive for the status of the target argument z. Since x is accepted, it suffices for the success of the attack $(\{x, y\}, z)$ to consider the status of y alone. Therefore, one can solely consider the remaining part of the attack in SF_2. Resulting from this, we obtain $SF_2^{\star} = (\{y, z\}, \{(y, z)\})$ (right). Given that $\{y\} \in adm(SF_2^{\star})$, we retrieve $\{x\} \cup \{y\}$ as an admissible set of SF.*



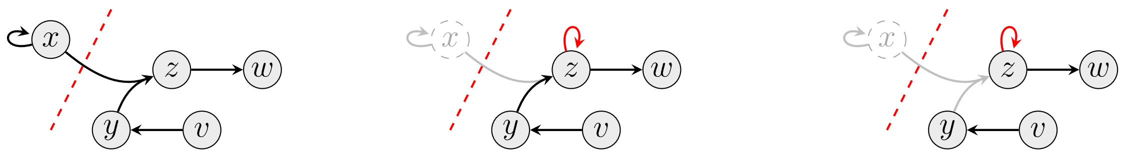
More generally, given a set of arguments E which is accepted in SF_1, the success of any link $(T, h) \in R_3$ where $T \cap A_1 \subseteq E$ is dependent only on the status of $T \setminus E$ in SF_2. Opposite considerations can be made when arguments in the links' tails are defeated (case (ii)).

Example 13. *Consider the following SETAF SF (left) with $E = \{w\} \in adm(SF_1)$. As in Example 10, the argument $w \in E$ attacks part of the link's tail (i.e. x), thereby neutralizing the collective attack $(\{x, y\}, z)$. As a result, w is compatible with both y and z. More formally, the rightmost part of SF is modified to obtain $SF_2^\star = (\{y, z\}, \emptyset)$ (middle). Indeed, their set-union $\{w\} \cup \{y, z\}$ is an admissible set of SF. Note that this case (ii) can also occur via a link. Consider the splitting for the SETAF $SF^\dagger$ (right) where $\{a, b\}$ is a preferred extension in the left sub-framework. Since a defeats c, the link $(\{b, c\}, d))$ has to be deleted in the modified framework, even though all tail-arguments of the link within A_1 (i.e., in our case b) are accepted.*



As before, we can directly exploit (ii) in order to guide the modification of SF_2 to get extensions for the whole SETAF. It is however less straightforward to find a correct modification for case (iii), where a link is undecided due to what happens in SF_1.

Example 14. *For SF (left), we have $E = \emptyset \in adm(SF_1)$ which means $x \in A_1 \setminus E^{\oplus}_{R_1}$, i.e., the link $(\{x, y\}, z)$* is undecided. *By naively applying the same technique as in the AF case (see Example 11), we make z self-attacking. However, it is not immediately clear whether one should modify SF_2 to include $(y, z) \in R_2^\star$ or not. It turns out that both options, i.e.,* (a) *including (y, z), and* (b) *not including (y, z) both lead to an undesired result.*



(a) (b)

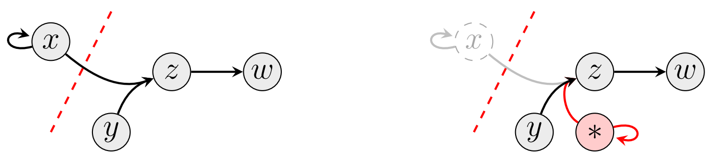
We see that $\{v\}$ is admissible in both case (a) *and* (b). *However, the additional self-loop (z, z) resulting from modification makes z not acceptable in both cases. This is in contrast with the fact that $\emptyset \cup \{v, z\}$ is indeed an admissible extension of SF.*

We see that a naive generalization of the AF approach, where we blindly make those arguments that are attacked by an undecided attack self-attacking, does not work as intended. In contrast to the AF case, these attacks can still be counter-attacked if in the second part of the framework an argument of the tail is attacked—as is the case in Example 14 where y is defeated by v. In this situation, z should remain acceptable. In particular, we have to ensure that (1) $\{z\}$ is conflict-free in the modified $SF_2^\star$, and (2) z can only be accepted if the remaining part of the attack $(\{x, y\}, z)$

is counter-attacked (e.g., in Example 14, if y is attacked). In fact, being *out*, the attack (y, z) is too weak and gets overwritten by the *undec* self-loop over z, thereby letting the status of z be entirely dependent on that of x. This is in opposition to what happens in the original SETAF, where the acceptance of z depends on the fact that y is defeated.

In order to present a splitting-based algorithm that works in a truly incremental and modular fashion, we consider a possible modification that is intermediate between adding an attack or not. For this, we have to make sure that the remaining part of the link is not "powerful" enough to actually defeat z—while at the same time indicating a need for z to defend against the remaining part of the attack. In the SCC-recursive schema for SETAFS [12] this issue has been resolved by marking certain attacks as *mitigated attacks*, i.e., those are attacks that have to be counter-attacked in order to accept the target argument, but cannot be used to attack the target argument in order to defend some other argument. In a nutshell, a conflict-free set is admissible if for each attack towards the set (mitigated or not) a non-mitigated counter-attack exists. However, in our modification-based approach we cannot add new syntactic concepts and adjusted semantics but have to encode this behavior within the standard SETAF syntax. A way to do this is by adding a self-attacking argument in the second sub-framework which participates in the collective attack along with the remaining part of it. This duplicate argument carries out the work of the undecided argument that is lost after splitting. Such modification can be visualized in the following example.

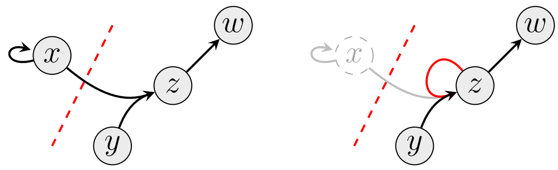
Example 15. *Contrary to the idea of mitigated attacks discussed before, we do not need labeling for attacks in this scenario. Instead, we add a dummy argument $*$ for the undecided attack $(\{x, y\}, z)$ such that $*$ is self-attacking and attacks z together with y (via the attack $(\{*, y\}, z)$).*



Modifying the second part of the framework in this way successfully neutralizes the acceptance of w, which is now faced by an undecided attack. Notably, the second part of the framework is identical to the whole SETAF prior to modification.

Such an addition can result in augmenting the number of arguments and attacks in the rightmost part of the framework. More importantly, such an unwanted outcome can be easily avoided. In fact, we can employ the very same target argument to do the job that was previously done by the dummy argument. As a consequence, the dummy argument becomes obsolete, and the modified attack collapses onto one singular (set)-self-attack on the target, as the next example illustrates.

Example 16. *As a final strategy we introduce a more concise and elegant modification of the second part of the framework at hand. Instead of using a dummy argument to make the attack towards z undecided, we choose to use z itself. This way, we obtain the expected result without creating a duplicate of x in the second part of the framework. This means, to account for the attack $(\{x, y\}, z)$ with the undecided argument x in the original framework (left), we introduce in the modification the attack $(\{y, z\}, z)$ which is a "set-self-attack" (right). Note that z is* not *a classic "self-attacker" as in the AF-case, since we do not introduce an attack (z, z).*



In the modification (right), the attack $(\{z, y\}, z)$ can never defeat its head z (since its conflicting tail $\{y, z\}$ would have to be accepted). Since z is not defeated, it is undecided, which carries over to w via the unaltered attack (z, w). This yields the admissible set $\emptyset \cup \{y\}$ as desired.

As a sanity check, note that this is indeed a generalization of the AF modification. In fact, for attacks (T, h) with $|T| = 1$, e.g. for $(T, h) = (\{t\}, h)$, we have $T \setminus \{t\} = \emptyset$. Thus, we add the self-attack $(\emptyset \cup \{t\}, t)$ which is the attack (t, t).

4 Reduct, SETAF Modification, and Splitting Theorem

In this section, we introduce the formal definitions that are needed to prove the correctness of our proposed splitting-based algorithm. Following [2], we generalize the notions of *reduct* and *modification*, in application to the rightmost part of the original SETAFs. Intuitively, the reduct takes care of the arguments in SF_2 that are already defeated by E_1 by removing them, and modifies the links that we characterize as *in*–case (1)—s.t. the remaining attack appears in the reduct SF_2'. The modification then "modifies" the undecided links by adding the targeted argument to the tail—we add (set-)self-attacks, as discussed in Example 16.

Hence, after computing an extension E_1 in SF_1, we obtain the reduct of SF_2 w.r.t. E_1 (i.e., SF_2') as follows:

1. We remove the arguments $a \in (E_1)^+_{R_3}$ which are defeated by E_1, together with their in- and outgoing attacks, which we realize by only keeping those attacks from R_2 which are completely within the new set of arguments A_2' (as in the original approach of Baumann),

2. we add the remaining part $T \cap A_2'$ of a link $(T, h) \in R_3$ if the tail arguments in $T \cap A_1$ are all in E_1 and no tail argument $t \in T$ is defeated via R_3 (as showcased in Example 12), given that there are any tail-arguments left in SF_2 (i.e., $T \cap A_2' \neq \emptyset$).

This allows us to retain all the information concerning defeated arguments and *in* attacks of SF_1. Formally this translates to the following notion of reduct (which we illustrate in Example 20):

Definition 17 (Reduct)**.** *Let (SF_1, SF_2, R_3) be a splitting for a SETAF SF. We define the (E_1, R_3)-*reduct *(or simply reduct) of SF_2 for some extension E_1 of SF_1 as the SETAF $SF_2' = (A_2', R_2')$ where, $A_2' = \{a \in A_2 \mid a \notin (E_1)^+_{R_3}\}$ and*

$$
\begin{aligned}
R_2' = &\{(T, h) \in R_2 \mid T \subseteq A_2', h \in A_2'\} \cup \\
&\{(T \cap A_2', h) \mid (T, h) \in R_3, T \cap A_2' \neq \emptyset, h \in A_2', T \cap A_1 \subseteq E_1, T \cap (E_1)^+_{R_3} = \emptyset\}
\end{aligned}
$$

We have argued throughout this paper that in fact, when dealing with undecidedness, what guided our intuition towards a certain modification is not the status of the arguments in SF_1, but rather the status of the links (corresponding to cases (i)-(iii)). In fact, if we closely examine Definition 5 we can see that even in the AF case we add self-attacks to those arguments that are targeted by an undecided *link*—the set U_E of undecided arguments is merely a tool to formally obtain those links. In the context of SETAFs, where an attack is not associated to exactly one attacker, this becomes even more evident. Hence, we decide to slightly tweak the definition to omit such detour, and base our notion solely on the undecided *links*.

Definition 18 (Undecided Links)**.** *Given a splitting (SF_1, SF_2, R_3) for a SETAF SF and an extension $E_1 \in SF_1$ we define the* set of undecided links *w.r.t. E_1 as:*

$$U^{E_1}_{R_3} = \{(T,h) \in R_3 \mid T \cap (E_1)^+_{R_1 \cup R_3} = \emptyset \text{ and } \exists t \in T : t \in A_1 \setminus (E_1)^{\oplus}_{R_1}\}$$

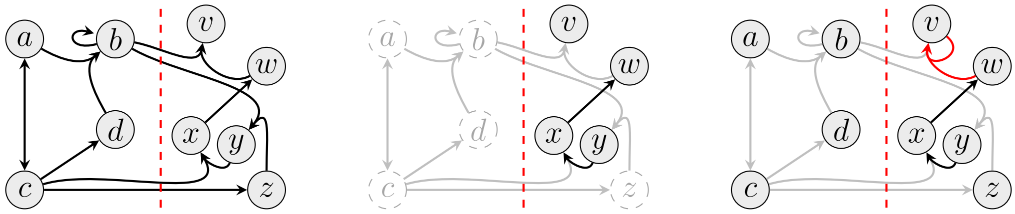
In what follows, we define the *modification*, which is applied on the reduct, and accounts for the effects of the undecided links. In particular, for each undecided link, we add to the targeted argument a (set-)self-attack incorporating the remaining part of the link (as intuitively explained in Example 16).

Definition 19 (Modification)**.** *Let (SF_1, SF_2, R_3) be a splitting for a SETAF SF and E_1 an extension of SF_1. Take SF'_2 as the (E_1, R_3)-reduct of SF_2 and $U^{E_1}_{R_3}$ as the set of undecided links w.r.t. E_1. We denote with $mod^{E_1}_{R_3}(SF'_2)$ the $U^{E_1}_{R_3}$-modification (or simply modification) of SF'_2 s.t.:*

$$mod^{E_1}_{R_3}(SF'_2) = (A'_2, R'_2 \cup \{((T \cap A'_2) \cup \{h\}, h) \mid (T,h) \in U^{E_1}_{R_3}, h \in A'_2\})$$

Before we present the main result of this paper we want to illustrate Definitions 17–19 in the following example, while covering many interesting cases at once.

Example 20. *In (a) we have a new SETAF SF with a splitting that separates the arguments $A_1 = \{a,b,c,d\}$ from $A_2 = \{v,w,x,y,z\}$. We see that $E_1 = \{c\}$ is admissible in the left part of the splitting. In (b) we see the reduct w.r.t. the set $\{c\}$, where a and d are defeated by c (as $\{c\}^+_{R_1} = \{a,d\}$) and b is undecided. This reduct contains from the right part all arguments except z, which is defeated by c (as $\{c\}^+_{R_3} = \{z\}$). We see that most attacks are removed from the right part, but (x,w) persists (since it is in R_2 and all involved arguments remain), and the attack $(\{c,y\},x)$ is changed to (y,x). The attack $(\{b,z\},y)$ is removed since z is defeated. The attack $(\{b,w\},v)$ is also removed, as b is undecided (i.e., $\{b,w\} \cap A_1 \not\subseteq E_1$). However, in (c) we see that the latter case is important for the modification: the attack $(\{b,w\},v)$ is an* undecided link*, which means in the modification we introduce the attack $(\{v,w\},v)$. For the right part of the splitting we see that $\{y,w\}$ is admissible, and obtain $\{c,y,w\}$ as an admissible set for SF.*



(a) SETAF SF *(b) $(\{c\}, R_3)$-reduct* *(c) $U^{\{c\}}_{R_3}$-modification*

Note that in the first step, for the left part of the splitting, instead of the set $\{c\}$ we could also investigate the admissible sets $\emptyset$, $\{a\}$, or $\{a, d\}$, which result in different reducts and modifications.

Having these notions at hand, we now establish the adequacy of our splitting technique for SETAFs. We start by establishing that (a) conflict-freeness of the sub-frameworks SF_1 and SF_2 carries over to the whole SETAF SF, and (b) conflict-free sets of SF induce conflict-free subsets in SF_1 and SF_2'.

Proposition 21. *Let (SF_1, SF_2, R_3) be a splitting for a SETAF $SF = (A, R)$ with $SF_1 = (A_1, R_1)$ and $SF_2 = (A_2, R_2)$. Let $SF_2^\star = mod_{R_3}^{E_1}(SF_2')$.*

1. *If $E_1 \in cf(SF_1)$ and $E_2 \in cf(SF_2^\star)$, then $E_1 \cup E_2 \in cf(SF)$.*
2. *If $E \in cf(SF)$, then $E \cap A_1 \in cf(SF_1)$ and $E \cap A_2 \in cf(SF_2')$.*

Proof. (1.) We need to show for each $(T, h) \in R_1 \cup R_2 \cup R_3$ that $T \cup \{h\} \not\subseteq E = E_1 \cup E_2$. Let $SF_2' = (A_2', R_2')$ and $SF_2^\star = (A_2^\star, R_2^\star)$. If $(T, h) \in R_1$ we immediately get $T \cup \{h\} \not\subseteq E$, since we know E_1 is conflict-free in SF_1. For $(T, h) \in R_2$ there are two cases: either (a) the attack is removed when we construct the reduct or (b) the attack remains, i.e., (T, h) in $SF_2^\star$. Case (a) happens if some $a \in T \cup \{h\}$ is attacked by E_1, i.e., $(T \cup \{h\}) \cap (E_1)^+_{R_1 \cup R_3} \neq \emptyset$. Then at least one argument $a \in T \cup \{h\}$ of the attack does not occur in the modification (i.e., $(T \cup \{h\}) \not\subseteq A_2^\star$), and since we assume $E_2 \in cf(SF_2^\star)$ we know $E_2 \subseteq A_2^\star$. Hence we obtain $T \cup \{h\} \not\subseteq E$. For case (b) we get from $E_2 \in cf(SF^\star)$ that at least one argument $a \in T \cup \{h\}$ is not in E_2, which also means $T \cup \{h\} \not\subseteq E$. Finally, for $(T, h) \in R_3$ we again consider two cases: (a) $T \cap A_1 \subseteq E_1$, and (b) $T \cap A_1 \not\subseteq E_1$. For case (a) we either have $T \subseteq A_1$ in which case $h \in (E_1)^+_{R_3}$ and we obtain $h \notin E$ (since then $h \notin A_2'$ while we know $E_2 \subseteq A_2'$), or if $T \not\subseteq A_1$ we get an attack $(T \cap A_2', h) \in R_2^\star$ (if otherwise $T \cap A_2' = \emptyset$ this means we removed some $a \in T \cap A_2$ when constructing the reduct, which means $a \notin E_2$ and consequently $a \notin E$), which since $E_2 \in cf(SF_2^\star)$ either means $T \cap A_2' \not\subseteq E_2$ or $h \notin E_2'$, both give us $T \cup \{h\} \not\subseteq E$. For case (b) we have $T \cap A_1 \not\subseteq E_1$, which means $T \cup \{h\} \not\subseteq E$.

(2.) Suppose now that $E \in cf(SF)$. From this we derive that $E \cap A_1 \in cf(SF_1)$ because every subset of a conflict-free set is also conflict-free. We now show that $E \cap A_2 \in cf(SF_2')$. Given that $E \in cf(SF)$, then for all $T \subseteq E \cap A_1$ and $a \in E \cap A_2$, we have $(T, a) \notin R_3$. Hence, no argument in E is deleted going from SF_2 to the reduct SF_2'. Thus, we conclude that $E \cap A_2 \subseteq A_2'$. Moreover, by $E \in cf(SF)$ we know for each $(T, h) \in R_2$ that $T \cup \{h\} \not\subseteq E$ which carries over to SF_2', since the attacks in R_2 may be removed, but are never changed. Finally, whenever for a link $(T, h) \in R_3$ with $T \cap A_1 \subseteq E$ we add an attack $(T \cap A_2', h) \in R_2'$ when constructing the reduct, we also obtain $(T \cap A_2') \cup \{h\} \not\subseteq E$ since otherwise $T \cup \{h\} \subseteq E$. Therefore, $E \cap A_2 \in cf(SF_2')$ concluding the proof. □

Finally, we are ready to characterize the splitting algorithm by proving the main theorem of this paper for the standard Dung semantics. In particular, we show that 1. if one computes an extension E_1 in SF_1, then applies the previously discussed reduct and modification, and obtains an extension E_2 of the remaining sub-framework, the set-union of the two indeed make for an extension of the whole framework SF. This characterizes the incremental computation of the extension E

by evaluating the two sub-frameworks. Conversely, we show that 2. if we project an arbitrary extension E of the whole framework SF to the sub-frameworks, we obtain extensions E_1 for SF_1 and E_2 for the (w.r.t. E_1)-modified version of SF_2. This result generalizes the corresponding result of AFs [2].

Due to space constraints we present proof details only for admissible semantics, which are prototypical for the other semantics. Details for the remaining semantics can be found in the appendix.

Theorem 22. *Let (SF_1, SF_2, R_3) be a splitting for a SETAF $SF = (A, R)$ with $SF_1 = (A_1, R_1)$, $SF_2 = (A_2, R_2)$, and $\sigma \in \{stb, adm, com, pref, grd\}$.*

1. *If $E_1 \in \sigma(SF_1)$ and $E_2 \in \sigma\big(mod_{R_3}^{E_1}(SF_2')\big)$, then $E_1 \cup E_2 \in \sigma(SF)$.*

2. *If $E \in \sigma(SF)$, then $E \cap A_1 \in \sigma(SF_1)$ and $E \cap A_2 \in \sigma\big(mod_{R_3}^{E\cap A_1}(SF_2')\big)$.*

Proof. **(admissible)**. (1.) Since admissibility implies conflict-freeness, we know from Proposition 21 that $E = E_1 \cup E_2 \in cf(SF)$. We need to show that E defends itself in SF, i.e. for all $a \in E$, if $(T, a) \in R_1 \cup R_2 \cup R_3$, then $(T', t) \in R_1 \cup R_2 \cup R_3$ for $T' \subseteq E$ and $t \in T$. Consider an argument $a \in E_1$. E_1 defends a from each attack in R_1 towards a since $E_1 \in adm(SF_1)$. Therefore, $E_1 \in adm(SF)$. Consider now an argument $a \in E_2$ and an arbitrary attack $(T, a) \in R_2 \cup R_3$ towards a. If $T \cap (E_1)^+_{R_1\cup R_3} \neq \emptyset$ we know a is defended (in SF) by E_1 against (T, a) and we are done, hence, we proceed with the assumption $T \cap (E_1)^+_{R_1\cup R_3} = \emptyset$. This means that either $(T \cap A_2', a) \in R_2^\star$ (via the reduct) or $((T \cap A_2') \cup \{a\}, a) \in R_2^\star$ (via the modification). Since $a \in E_2$ and $E_2 \in adm(SF_2^\star)$ we know there is a counter-attack in $R_2^\star$ which defends a. Even in case $((T \cap A_2') \cup \{a\}, a) \in R_2^\star$ this counter-attack cannot be against a since this violates conflict-freeness of E_2 in $SF_2^\star$. Hence, there is some $(S, t) \in R_2^\star$ s.t. $S \subseteq E_2$ and $t \in T \cap A_2'$ with $t \notin S$. Hence, either (a) $(S, t) \in R_2$ in which case a is defended by E in SF or (b) there is some $(S', t) \in R_3$ with $S' \supset S$ s.t. $S' \cap A_1 \subseteq E_1$, in which case a is defended (in SF) by E via the attack (S', t) since then $S' \subseteq E_1 \cup E_2$. In any case we showed that a is defended in SF by E, i.e., $E \in adm(SF)$.

(2.) By Proposition 21 we get $E_1 = E \cap A_1 \in cf(SF_1)$ and $E_2 = E \cap A_2 \in cf(SF_2')$. Since E is defends itself in SF we get $E \cap A_1 \in adm(SF_1)$ because (SF_1, SF_2, R_3) is a splitting of SF, i.e. no argument in $E \cap A_1$ is attacked by a subset of A_2 or defended by $E \cap A_2$. That is, in SF_1 every attack towards an argument in $E \cap A_1$ is countered by $E \cap A_1$. It remains to show that $E \cap A_2 \in adm(SF_2^\star)$. Consider now an argument $a \in E_2$ and an arbitrary attack $(T, a) \in R_2^\star$ against a. This attack (T, a) either corresponds to an attack $(T, a) \in R_2$ or $(T', a) \in R_3$ with $T' \supset T \setminus \{a\}$ (which accounts for both the case of addition in the reduct and the modification). In both cases we have that $T \cap (E_1)^+_{R_1\cup R_3} = \emptyset$ (or $T' \cap (E_1)^+_{R_1\cup R_3} = \emptyset$, resp.) as otherwise (T, a) would not be in $R_2^\star$. However, since a is defended by E in SF, there is a counter-attack $(S, t) \in R_2 \cup R_3$ s.t. $S \subseteq E$ and $t \in (T \setminus \{a\})$ (or $t \in (T' \setminus \{a\})$, resp.). If $(S, t) \in R_2$ then from $S \subseteq E$ and $E_2 \subseteq A_2'$ (which we get from $E_2 \in cf(SF_2')$ via Proposition 21) and the fact that then $(S, t) \in R_2'$ since $S \cup \{t\} \subseteq A_2'$ we get that E_2 defends a via (S, t) against (T, a) in $SF_2^\star$. If $(S, t) \in R_3$ since $S \subseteq E$ we have $S \cap A_1 \subseteq E_1$, and hence we get an attack $(S \cap A_2', t) \in R_2'$

which again defends a against (T, a) in $SF_2^\star$. Hence, in every case a is defended in $SF_2^\star$, i.e., $E_2 \in adm(SF_2^\star)$. □

To further establish the adequacy of our splitting approach for SETAFs, we want to highlight that we retain the close connection to the *directionality principle* [12] already proven for AFs [3]. Before introducing the definition of directionality, we first recall the notion of *influence*: in a SETAF $SF = (A, R)$ an argument $a \in A$ *influences* an argument $b \in A$ if there is a path $(a, p_1, \ldots, p_n, b)$ in SF s.t. for $1 \leq i < n$, $(T_i, p_{i+1}) \in R$ with $p_i \in T_i$, as well as $(T_0, p_1), (T_n, b) \in R$ with $a \in T_0$, $p_n \in T_n$ holds (i.e., there is a path from a to b if we "break up" the hyperedges to standard directed edges—the *primal graph* of SF). A set $S \subseteq A$ is an *uninfluenced set* in SF (denoted $S \in US(SF)$) if no $a \in A \setminus S$ influences any $b \in S$. In other words, a set S is uninfluenced in SF if it has no incoming edges. In a nutshell, directionality states that the *projection* $SF{\downarrow_U}$ of a SETAF SF to an uninfluenced set U yields the same extensions as the original framework (ignoring the arguments removed under projection).

Definition 23. *A semantics σ satisfies* directionality *if for all SETAFs SF and every $U \in US(SF)$ it holds $\sigma(SF{\downarrow_U}) = \{E \cap U \mid E \in \sigma(SF)\}$, where $SF{\downarrow_U} = (U, \{(T', h) \mid (T, h) \in R, h \in U, T' = T \cap U, T' \neq \emptyset\})$.*

We are now able to generalize the following result regarding directionality from AFs [3, Theorem 4.13].

Theorem 24. *Let σ be a semantics s.t. $|\sigma(SF)| \geq 1$ for each SETAF SF. If σ allows splitting (i.e., Theorem 22 holds for σ) then σ satisfies directionality.*

Proof. Assume towards contradiction this is not the case, i.e., for some SETAF $SF = (A, R)$ and some $U \in US(SF)$ it holds $\sigma(SF{\downarrow_U}) \neq \{E \cap U \mid E \in \sigma(SF)\}$. Observe that $(SF{\downarrow_U}, SF_2, R_3)$ is a splitting of SF, where $SF_2 = (A \setminus U, R \cap (2^{A \setminus U} \times (A \setminus U)))$ and R_3 contains exactly those attacks of SF that are neither in $SF{\downarrow_U}$ nor SF_2.

($\not\subseteq$): This means there is some $E_1 \in \sigma(SF{\downarrow_U})$ s.t. $E_1 \neq \{E \cap U\}$ for any $E \in \sigma(SF)$. By Theorem 22 and since $|\sigma(SF_2^\star)| \geq 1$ we get $E_2 \in \sigma(SF_2^\star)$, where $SF_2^\star = mod_{R_3}^{E_1}(SF_2')$ and SF_2' is the (E_1, R_3)-reduct of SF_2. Then by Theorem 22 we get $E_1 \cup E_2 \in \sigma(SF)$, a contradiction to the assumption that there is no $E \in \sigma(SF)$ s.t. $E_1 = E \cap U$.

($\not\supseteq$): This means there is some $E \in \sigma(SF)$ s.t. there is no $E_1 \in \sigma(SF{\downarrow_U})$ with $E_1 = \{E \cap U\}$, directly contradicting Theorem 22. □

5 Discussion

In this paper, we introduced a modification-based splitting approach for SETAFs, and showed that it generalizes the important key features of its AF counterpart. In the following, we clarify the relation of our splitting approach to SCC-recursiveness (as due to [12] for SETAFs) and splitting for abstract dialectal frameworks (ADFs) [16]. In the incremental computation approach induced by the SCC-recursive property, one computes the extensions in subframeworks of a given SETAF, and ultimately combines the thereby computed extension parts (as in the splitting approach). In contrast to splittings however, this is restricted to subframeworks that make up *strongly connected components* w.r.t. the primal graph of the SETAF. Splitting on the other hand is more general in this regard, as the subframeworks do not have to be strongly connected. Finally, SCC-recursiveness relies on a generalized semantics to deal with the decisions of prior parts of the framework, in contrast to the syntactic manipulation-based approach of splitting.

ADFs [6] are an expressive argumentation formalism, where each argument is associated with a propositional formula over arguments as variables as an acceptance condition. It is well-known that SETAFs can be interpreted as a special kind of ADFs with acceptance conditions in the form of a conjunction of disjunctive clauses of negated literals [11]. That is, in principle we can apply ADF splitting to SETAFs. However, it is not clear that following the ADF approach the modified second framework again is of the desired (SETAF-like) form and whether one can avoid certain overheads in the simpler case of SETAFs. Upon closer inspection and with minor syntactic manipulation the ADF approach in the special case of SETAF-like frameworks is similar to what we discussed in Example 15, where we introduced artificial "dummy"-arguments. However, such a trick is not needed in our case, as we have illustrated.

In summary, we showed how the splitting technique can be applied in the context of collective attacks, where in contrast to the AF case also intricate situations like "diagonal splitting" can occur. We furthermore showed that the splitting theorem holds in the setting of SETAFs, and established that we retain the strong link to directionality which is known for AFs.

Our result can serve as a starting point for more general splitting ideas like parameterized splitting (cf. [4] for AFs), as well as a broader consideration in the context of dynamic argumentation. Future work includes the generalization to parameterized splitting also for SETAFs, as well as an implementation of the algorithm.

Acknowledgments

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A Proof of Theorem 22

Theorem 22. *Let (SF_1, SF_2, R_3) be a splitting for a SETAF $SF = (A, R)$ with $SF_1 = (A_1, R_1)$, $SF_2 = (A_2, R_2)$, and $\sigma \in \{stb, adm, com, pref, grd\}$.*

1. *If $E_1 \in \sigma(SF_1)$ and $E_2 \in \sigma\big(mod_{R_3}^{E_1}(SF_2')\big)$, then $E_1 \cup E_2 \in \sigma(SF)$.*
2. *If $E \in \sigma(SF)$, then $E \cap A_1 \in \sigma(SF_1)$ and $E \cap A_2 \in \sigma\big(mod_{R_3}^{E\cap A_1}(SF_2')\big)$.*

Proof. In what follows, we prove 1. and 2. for each semantics. For notational convenience, let $E = E_1 \cup E_2$ and let $SF_2' = (A', R')$ be the reduct of SF_2 w.r.t. $E_1 = E \cap A_1$, and $SF_2^\star = (A^\star, R^\star) = mod_{R_3}^{E_1}(SF_2')$ be the modification.

(stable). (1.) From Proposition 21 together with the assumptions that $E_1 \in stb(SF_1)$ and $E_2 \in stb(SF_2^\star)$, we know that $E = E_1 \cup E_2 \in cf(SF)$. Let $a \in A \setminus E$, we show that $a \in E_R^+$. If $a \in A_1$ by $E_1 \in stb(SF_1)$ we get $a \in (E_1)_{R_1}^+$, which immediately gives us $a \in E_R^+$. If $a \in A_2$, either $a \in A' = A^\star$ or $a \notin A' = A^\star$. If $a \notin A' = A^\star$ this can only be because $a \in (E_1)_{R_3}^+$ which gives us $a \in E_R^+$. If $a \in A' = A^\star$ then by $E_2 \in stb(SF^\star)$ we get $a \in (E_2)_{R_2^\star}^+$. Hence, either E_2 defeats (in $SF_2^\star$) a via some $(T, a) \in R_2$ (in which case $a \in E_R^+$) or via some other $(T, a) \in R_2^\star \setminus R_2$, which can only be the remaining part of an attack from R_3. Clearly in this case $a \notin T$ (as then (T, a) would not defeat a), so we know (T, a) is not constructed from an undecided link in $U_{R_3}^{E_1}$ (which do not occur in stable semantics as $A_1 = E_1 \cup (E_1)_{R_1}^+$). Instead, we must have obtained (T, a) while constructing the reduct, i.e., there is an attack $(T', a) \in R_3$ with $T' \supset T$, and $T' \cap A_1 \subseteq E_1$. From this we get $T' \subseteq E$, and consequently $a \in E_{R_3}^+$. In all cases we get $a \in E_R^+$, which means $A = E \cup E^+$, i.e., $E \in stb(SF)$.

(2.) Assume $E \in stb(SF)$. From this we know that $E_R^\oplus = A = A_1 \cup A_2$. We first prove that $E_1 = E \cap A_1 \in stb(SF_1)$. From Proposition 21 we know $E \cap A_1 \in cf(SF_1)$. Since (SF_1, SF_2, R_3) is a splitting of SF we know that the only attacks towards arguments in A_1 are from R_1, so we immediately get $(E \cap A_1)_{R_1}^\oplus = A_1$, i.e., $E_1 \in stb(SF_1)$. We know turn to prove $E_2 = E \cap A_2 \in stb(SF_2^\star)$. First, notice that $SF_2^\star = SF_2'$ because $U_{R_3}^{E_1} = \emptyset$. From Proposition 21 we again obtain $E_2 \in cf(SF_2')$. Let $a \in A_2' \setminus E_2$. We show $a \in (E_2)_{R_2'}^+$. Since $E \in stb(SF)$ we know $a \in E_R^+$ which means (a) $a \in E_{R_2}^+$ or (b) $a \in E_{R_3}^+$. In case (a) we have an attack $(T, a) \in R_2$ with $T \subseteq E_2$, and since $E_2 \subseteq A_2'$ (which is because $E_2 \in cf(SF')$) and also $a \in A_2'$ by assumption we know $(T, a) \in R_2'$, i.e., $a \in (E_2)_{R_2'}^+$. If (b) is the case we know that there is some $(T, a) \in R_3$ with $T \subseteq E$, i.e., $T \cap A_1 \subseteq E_1$. Clearly since E is conflict-free in SF we have $T \cap (E_1)_{R_1 \cup R_3}^+ = \emptyset$ which means in SF_2' we have an attack $(T \cap A_2', a)$. Since $T \cap A_2' \subseteq E_2$ we get $a \in (E_2)_{R_2'}^+$. In both cases we get $E_2 \cup (E_2)_{R_2'}^+ = A_2'$, i.e., $E_2 \in stb(SF_2') = stb(SF_2^\star)$.

(complete). (1.) Given statement 1 of admissible semantics proven above, we only need to show that $a \in E_1 \cup E_2$ for all $a \in A$ defended by $E_1 \cup E_2$ in SF. Assume towards contradiction that there is an $a \in (A_1 \cup A_2) \setminus (E_1 \cup E_2)$ defended by $E_1 \cup E_2$. From $E_1 \in com(SF_1)$, we know that $a \notin A_1 \setminus E_1$. Hence, $a \in A_2 \setminus E_2$ and, because (SF_1, SF_2, R_3) is a splitting and $E_1 \in cf(SF_1)$, we obtain $a \in A_2' \setminus E_2$. Indeed, if $a \in (E_1)_{R_3}^+$, then $E_1 \cup E_2$ defends a from an attack of E_1, which is against conflict-freeness of $E_1 \cup E_2$. Consider now possible attacks scenarios towards a: if a is not attacked, then it would be in every complete extension, hence $E_2 \notin com(SF_2^\star)$. If a is attacked by some set of arguments T, then $(T, a) \in R_2$ or $(T, a) \in R_3$. We show that both cases lead to a contradiction. Consider now $(T, a) \in R_2$. Again, in this case we distinguish three attack scenarios:

1. $(T, a) \in R_2$ with $T \cap (E_1)^+_{R_1 \cup R_3} \neq \emptyset$. Since a is defended by $E_1 \cup E_2$, such attacks are countered by E_1 via a link. Given that $(T, a) \notin R^\star_2$ (eliminated by the reduct), a is vacuously defended by E_2 in $SF^\star_2$. Thus, $E_2 \notin com(SF^\star_2)$.

2. $(T, a) \in R_2 \setminus \{(a, a)\}$ with $T \cap (E_1)^+_{R_1 \cup R_3} = \emptyset$. This means there is a counter-attack $(S, t) \in R_2 \cup R_3$ with $S \subseteq E_1 \cup E_2$. Given that the reduct and modification do not eliminate such attacks, a is defended by E_2 in $SF^\star_2$. Thus, $E_2 \notin com(SF^\star_2)$.

3. $(T, a) = (a, a) \in R_2$ with $T \cap (E_1)^+_{R_1 \cup R_3} = \emptyset$. From SETAF Fundamental Lemma [17] and the assumption that a is defended by $E_1 \cup E_2$, we get that $E_1 \cup E_2 \cup \{a\}$ is an admissible (and thus conflict-free) extension in SF. Since a is a self-attacking argument, we derive a contradiction.

All of the above derive a contradiction. Therefore, we now consider the case where $(T, a) \in R_3$. Since $E_1 \cup E_2$ defends a, we know that for some $t \in T \cap A_1$, $(E_1, t) \in R_1$ or for some $t \in T \cap A_2$ and $S \subseteq E_1 \cup E_2$, $(S, t) \in R_3$. In the first case, the reduct of SF_2 does not contain $(T \cap A'_2, a)$ because $T \cap (E_1)^+_{R_1} \neq \emptyset$. Hence, a is unattacked in R'_2. For the same reason, and given that $(E_1)^+_{R_1 \cup R_3} \supseteq (E_1)^+_{R_1}$, we also know that $T \cap (E_1)^+_{R_1 \cup R_3} \neq \emptyset$. Thus, $(T, a) \notin U^{E_1}_{R_3}$ and a is unattacked in $R^\star_2$. Again, a is vacuously defended by E_2 in $SF^\star_2$ and $E_2 \notin com(SF^\star_2)$. Contradiction. Consider now the case where $(S, t) \in R_3$ for some $S \subseteq E_1 \cup E_2$ and $t \in T \cap A_2$. if $S \subseteq E_1$, then $T \cap (E_1)^+_{R_3} \neq \emptyset$ and $(T \cap A'_2, a) \notin R'_2$. For the same reason as before, $(T \cap A'_2, a) \notin R^\star_2$. Hence, a is vacuously defended by E_2 in $SF^\star_2$ and $E_2 \notin com(SF^\star_2)$. If $S \not\subseteq E_1$, then $(S \cap A'_2, t) \in R'_2$ because $S \cap A'_2 \neq \emptyset$, $t \in A'_2$, $S \cap A_1 \subseteq E_1$ and $S \cap (E_1)^+_{R_1 \cup R_3} = \emptyset$. We derive directly that $(S \cap A'_2, t) \in R^\star_2$ since the modification does not delete attacks. Hence, E_2 defends a in $SF^\star_2$. Finally, this contradicts our hypothesis that $E_2 \in com(SF^\star_2)$, concluding the proof.

(2.) Admissibility of $E_1 = E \cap A_1$ and $E_2 = E \cap A_2$ has been shown above. We need to show that for all a defended by E_1 in SF_1 and by E_2 in $SF^\star_2$, we have $a \in E_1$ and $a \in E_2$ respectively. Let us consider E_1 first. Towards contradiction, assume there is an $a \in A_1 \setminus E_1$ such that a is defended by E_1. This implies that a is such that $a \in A_1 \cup A_2 \setminus E$ and a is defended by E, in contradiction with the completeness of E in SF. Consider now E_2. As before, we need to show that there is no $a \in A'_2 \setminus E_2$ such that E_2 defends a in $SF^\star_2$. Again, assume that there is such an $a \in A'_2 \setminus E_2$. As before, we consider possible attack scenarios towards a. If a does not receive any attack in $SF^\star_2$, then for every attack $(T, a) \in R_2 \cup R_3$ it holds that $T \cap (E_1)^+_{R_1 \cup R_3} \neq \emptyset$ ((T, a) was eliminated by the reduct). Hence, E_1 defends a in SF, in contradiction with $E \in com(SF)$. Assume now that a receives an attack in $SF^\star_2$, then for all $(T, a) \in R^\star_2$ we have $(S, t) \in R^\star_2$ for some $S \subseteq E_2$ and $t \in T$, as we assume that E_2 defends a in $R^\star_2$. Note that $t \notin S$, as otherwise (S, t) would not counter the attack (T, a) which we assumed. But we know $(S, t) \in R^\star_2$ corresponds to some attack in $R_2 \cup R_3$. If $(S, t) \in R_2$ we have that E defends a in SF (via (S, t)), a contradiction to $E \in com(SF)$. If on the other hand there is some $(S', t) \in R_3$ with $S' \supset S$ we know that also $S' \cap A_1 \subseteq E_1$, as otherwise we would have $t \in S$ (if (S, t) was introduced in via the modification), which we already ruled out. Finally, the attack (T, a) is either in R_2 or corresponds to some attack $(T', a) \in R_3$ with $T' \subseteq T \setminus \{a\}$, in both cases SF defends a via (S, t) or (S', t) against the attack. Hence, we derive a contradiction to $E \in \mathit{com}(SF)$. As every possible way a could be defended by E_2 in $SF_2^\star$ but not in E_2 leads to a contradiction, this cannot be the case, hence, $E_2 \in \mathit{com}(SF_2^\star)$.

(preferred). (1) From statement 1 for admissible semantics above, we derive that $E_1 \cup E_2 \in \mathit{adm}(SF)$. Moreover, from hypothesis we have that there is no $S_1 \in \mathit{adm}(SF_1)$ such that $S_1 \supset E_1$ and no $S_2 \in \mathit{adm}(SF_2^\star)$ such that $S_2 \supset E_2$. We need to prove that there is no $S \in \mathit{adm}(SF)$ such that $S \supset E = E_1 \cup E_2$. Towards contradiction, suppose there is such an S. Then $S_1 = S \cap A_1 \supset E_1$ or $S_2 = S \cap A_2 \supset E_2$. Consider the first case. Since $E_1 \in \mathit{pref}(SF_1)$ by hypothesis, it must hold that $S_1 \notin \mathit{adm}(SF_1)$. However, this is in contradiction with statement 2 shown above for the admissible semantics (i.e. if $S \in \mathit{adm}(SF)$ and (SF_1, SF_2, R_3) is a splitting for SF, then $S \cap A_1 \in \mathit{adm}(SF_1)$). Consider now the case where $S_2 \supset E_2$. We can assume $S \cap A_1 = E_1$, as otherwise we derive a contradiction as above. Similarly to the case before, it must hold that $S_2 \notin \mathit{adm}(SF_2^\star)$. Again, since we assumed $S \in \mathit{adm}(SF)$, then it must hold that $S_2 \in \mathit{adm}(SF_2^\star)$ (statement 2 of admissible semantics). Both directions lead to a contradiction, hence there is no $S \in \mathit{adm}(SF)$ such that $S \supset E_1 \cup E_2$. Thus we conclude that $E_1 \cup E_2 \in \mathit{pref}(SF)$.

(2.) By hypothesis, we have $E \in \mathit{pref}(SF)$ and hence, there is no $S \in \mathit{adm}(SF)$ such that $S \supset E$. Moreover, by statement 2 of admissible semantics, we get $E \cap A_1 \in \mathit{adm}(SF_1)$ and $E \cap A_2 \in \mathit{adm}(SF_2^\star)$. Consider now $E \cap A_1$. By directionality of preferred semantics [12] we obtain $E \cap A_1 \in \mathit{pref}(SF_1)$. For $E \cap A_2$, assume now there is an $S_2 \in \mathit{adm}(SF_2^\star)$ such that $S_2 \supset E \cap A_2$. For statement 1 of admissible semantics, $(E \cap A_1) \cup S_2$ is admissible in SF which contradicts the maximality of E. This conclude the proof.

(grounded). (1) Since the grounded extension is also complete, we only need to show that $E_1 \cup E_2$ is the minimal complete extension in SF. Suppose the contrary is true: there is a set $S \in \mathit{com}(SF)$ such that $S \subset E_1 \cup E_2$. Hence, $S_1 = S \cap A_1 \subset E_1$ or $S_2 = S \cap A_2 \subset E_2$. Consider the first case. From the statement 2 of complete semantics above, we derive that $S_1 \in \mathit{com}(SF_1)$. But this contradicts our hypothesis that $E_1 \in \mathit{grd}(SF_1)$, since S_1 would be the $\subseteq$-minimal complete extension of SF_1. Hence, $S_1 = E_1$. For the second case, we can assume $S \cap A_1 = E_1$ (as otherwise we derive a contradiction via the first case) and we deduce again from statement 2 of complete semantics that $S_2 \in \mathit{com}(SF_2^\star)$. However, by hypothesis we have that $E_2 = \mathit{grd}(SF_2^\star)$, which is incompatible with that fact that $S_2 \subset E_2$. Both cases lead to a contradiction, hence we derive that $E_1 \cup E_2 = \mathit{grd}(SF)$.

(2) By hypothesis, we have $E \in \mathit{grd}(SF)$ and hence, there is no $S \in \mathit{adm}(SF)$ such that $S \subset E$. Moreover, by statement 2 of complete semantics, we get $E \cap A_1 \in \mathit{com}(SF_1)$ and $E \cap A_2 \in \mathit{com}(SF_2^\star)$. Consider now $E \cap A_1$. By directionality of grounded semantics [12] we obtain $E \cap A_1 \in \mathit{grd}(SF_1)$. For $E \cap A_2$, assume now there is an $S_2 \in \mathit{com}(SF_2^\star)$ such that $S_2 \subset E \cap A_2$. For statement 1 of complete semantics, $(E \cap A_1) \cup S_2$ is complete in SF which contradicts the minimality of E. This conclude the proof. □