Algorithmic Metatheorems for Second-Order Logic

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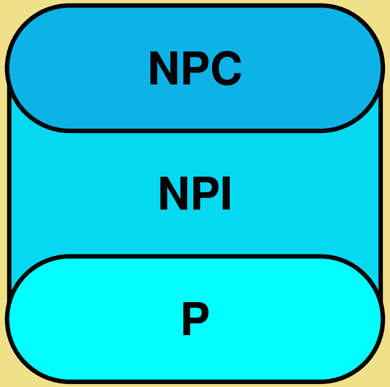
Starting Point

Fagin's Theorem

- A set of finite structures is in **NP** if and only it can be described by a formula of the form $\exists \boldsymbol{X_1} \cdots \exists \boldsymbol{X_m} \varphi$, where φ is a first-order formula ☞ ESO-logic

- Let us assume **P** $\neq$ **NP** for the remainder of this talk

- Then **NP** looks like this: ☞ Ladner's Theorem



- Therefore:
  - Some ESO-formulas describe **NP**-complete problems
  - Some ESO-formulas describe **NP**-intermediate problems
  - Some ESO-formulas describe problems in **P**

ESO: Example Formulas

- $\varphi_1 \stackrel{\text{def}}{=} \exists P \exists x_1 \exists x_2 \forall y_1 \forall y_2 \big(P(x_1) \wedge \neg P(x_2) \wedge$
  $(P(y_1) \wedge \neg P(y_2) \to (\neg E(y_1, y_2) \wedge \neg E(y_2, y_1))\big)$

- $\varphi_2 \stackrel{\text{def}}{=} \exists P_1 \exists P_2 \forall y_1 \forall y_2$
  $$\big(E(y_1, y_2) \to \bigwedge_{i=0,1,2} (\neg(\theta_i(y_1) \wedge \theta_i(y_2))\big),$$
  where
  - $\theta_0(x) \stackrel{\text{def}}{=} \neg P_1(x) \wedge \neg P_2(x)$
  - $\theta_1(x) \stackrel{\text{def}}{=} (P_1(x) \wedge \neg P_2(x)) \vee (\neg P_1(x) \wedge P_2(x))$
  - $\theta_2(x) \stackrel{\text{def}}{=} P_1(x) \wedge P_2(x)$

- $\varphi_3 \stackrel{\text{def}}{=} \exists T \forall x \exists y \big((E(x,x) \wedge E(x,y) \wedge \neg E(y,y) \wedge T(y)) \vee$
  $(\neg E(x,x) \wedge E(x,y) \wedge \neg E(y,y) \wedge \neg(T(x) \leftrightarrow T(y))\big)$

- $\varphi_4 \stackrel{\text{def}}{=} \exists P_1 \exists P_2 \forall x \exists y \big(E(x,y) \wedge$
  $((\theta_0(x) \wedge \theta_1(y)) \vee (\theta_1(x) \wedge \theta_2(y)) \vee (\theta_2(x) \wedge \theta_0(y)))\big)$

- $\varphi_5 \stackrel{\text{def}}{=} \exists R \forall x \exists y_1 \exists y_2 \forall z_1 \forall z_2 \forall z_3 \big(R(x, y_1) \wedge R(y_2, x) \wedge$
  $(R(x, z_1) \to z_1 = y_1) \wedge (R(z_1, x) \to z_1 = y_2) \wedge$
  $(((R(x, z_1) \wedge R(z_2, z_3)) \to (E(x, z_2) \leftrightarrow E(z_1, z_3)))\big)$

Obvious Question

- Given an ESO-formula φ, can we judge automatically whether it describes
  - an **NP**-complete problem,
  - an **NP**-intermediate problem,
  - a problem in **P**?

Digression

- Georg has a passion for *Schüttlers*

Schüttler 1

A **Schüttler** *consists of a rhyme with a twist*

and should be more fun than a twine round the wrist

Obvious Answer

- ...can we judge automatically...?
- Of course, not! ☞ undecidable
- If we cannot solve a problem exactly, we can go for approximate solutions
- Let us consider syntactically defined subclasses of ESO and classify them with respect to **NPC**, **NPI**, **P**
- Natural choice: formulas in prenex form with a quantifier prefix of a certain type
  - like for the classical Decision Problem

- **Notation:** We denote fragments by **prefix strings**, i.e., strings over $\{E, E_k, E^*, E_i^*, e, a \mid i \geqslant 1\}$, where
  - E_k, E: represent existential quantification of a k-ary (arbitrary) relation
  - E_k^*, E^*: represent existential quantification of an arbitrary number of k-ary (arbitrary) relations
  - e: represents existential first-order quantification
  - a: represents universal first-order quantification

- $\varphi_1 = \exists P \exists x_1 \exists x_2 \forall y_1 \forall y_2 \big(P(x_1) \land \neg P(x_2) \land (P(y_1) \land \neg P(y_2) \rightarrow (\neg E(y_1, y_2) \land \neg E(y_2, y_1))\big)$ is of the form
  - $E_1 eeaa$, but also $E_1 e^* a$ and $E e^* a^*$

- $\varphi_2 = \exists P_1 \exists P_2 \forall y_1 \forall y_2$
$$\big(E(y_1, y_2) \rightarrow \bigwedge_{i=0,1,2} (\neg(\theta_i(y_1) \land \theta_i(y_2))\big)$$
is of the form
  - $E_1 E_1 aa$, but also $E_1^* aa$

- Thus we want to know what kinds of problems can be expressed in fragments like $E_1 e^* a$ or $E^* ae$

Immediate Insight

- For each signature there is a **finite set** S of prefix strings such that for each prefix string s it holds
  - $s \not\leqslant S \Rightarrow$
    ESO(s) can describe **NP**-complete problems
    ☞ ESO(s) $\stackrel{\text{def}}{=}$ all ESO formulas with prefix string s
  - $s \leqslant S \Rightarrow$
    ESO(s) can **not** describe **NP**-complete problems

- $s \leqslant S \stackrel{\text{def}}{\Leftrightarrow} s$ is a subprefix string of a string $t \in S$

- For the classical Decision Problem this is known as Gurevich's Classification Theorem
  - relies on well-quasi-orders

- Questions:
  - Can we compute S?
  - Can we say more in case "$s \leqslant S$"?

“New” Result (General Form)

- For each of strings, directed graphs, undirected graphs there is a finite set S of prefix strings such that for each prefix string s it holds
  - $s \not\leqslant S \Rightarrow \mathrm{ESO}(s)$ can describe **NP**-complete problems
  - $s \leqslant S \Rightarrow \mathrm{ESO}(s)$ only describes problems in **P** (and even only regular sets in the case of strings)

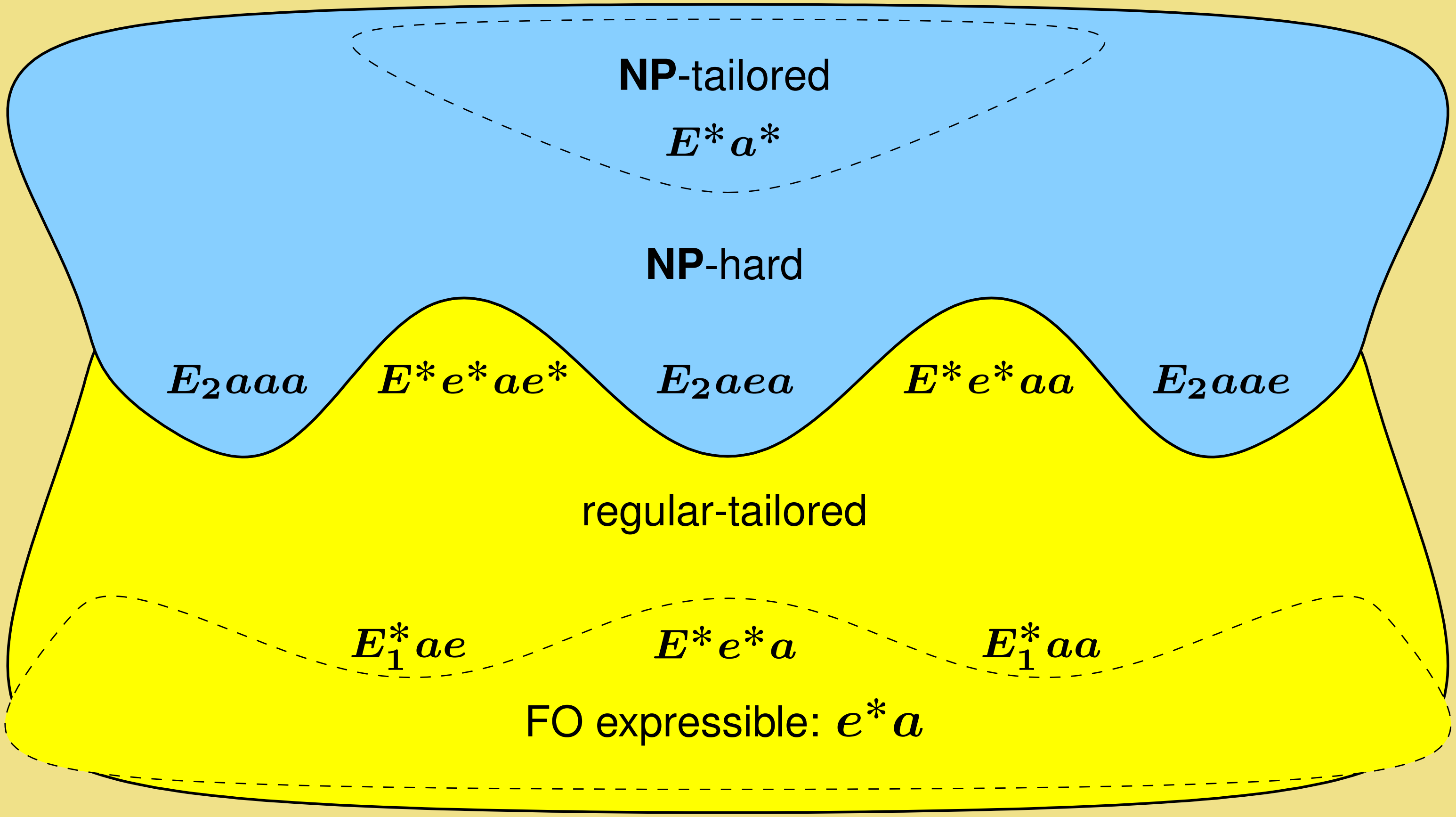
☞ End of story for **NPI**-problems!

- Furthermore: we do know S

Contents

ESO on Strings: Results

- The "critical set" S for strings was determined in [Eiter, Gottlob, Gurevich 2000]
- Furthermore the landscape was clarified as follows:



- Note: Strings have successor relation but not a linear order!

ESO on Strings: Lower Bounds

- The **NP**-hardness results are by reduction from 3SAT

- Propositional variables are encoded by 0-1-strings
- A propositional formula
  $$\chi = (p_0 \vee p_1 \vee \neg p_2) \wedge (\neg p_0 \vee \neg p_1 \vee p_2)$$
  is encoded as
  $$[(\mathbf{00}) + (\mathbf{01}) + (\mathbf{10}) -][(\mathbf{00}) - (\mathbf{01}) - (\mathbf{10}) +]$$

- A formula of the form $\exists V \exists G \exists R \exists R' \forall y_1 \forall y_2 \forall y_3 \psi$ can check whether χ is satisfiable:
  - V (unary) represents a truth value for each occurrence of a variable
  - G (unary) checks that each clause is satisfied by V
  - R (binary) checks that V is consistent (with the help of binary R')
    - R connects all pairs of identical prefixes of variable numbers...
- Then V, G, R' are eliminated...

ESO on Strings: Upper Bounds

- E^*e^*aa is regular
- In a nutshell...

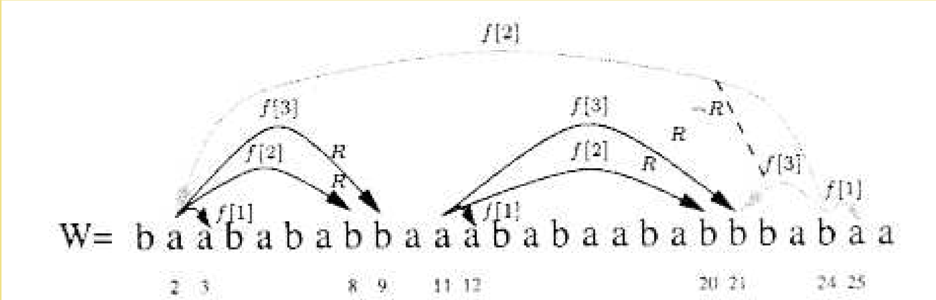
- Let $\exists R \forall y_1 \forall y_2 \psi$ be a formula ☞ R binary
- Whether $\psi(u, v)$ and $\psi(v, u)$ hold for positions $u \neq v$ depends on
  - $R(u, u)$
  - $R(v, v)$
  - the symbols at u and v
  - whether u and v are neighbours
  - $R(u, v)$
  - $R(v, u)$
- The choice of $R(u, v)$ and $R(v, u)$ does not affect any other pairs

➨ Atoms $R(u, v)$ and $R(v, u)$ can be eliminated

➨ $E_1^*e^*aa$

➨ regular

- $E^*e^*ae^*$ is regular
- This is a 23 page proof using concepts such as
  - hypergraph traversals
  - transducers and
  - four normal forms
- In a nutshell, it is shown that only a constant number of *remote witnesses* is needed

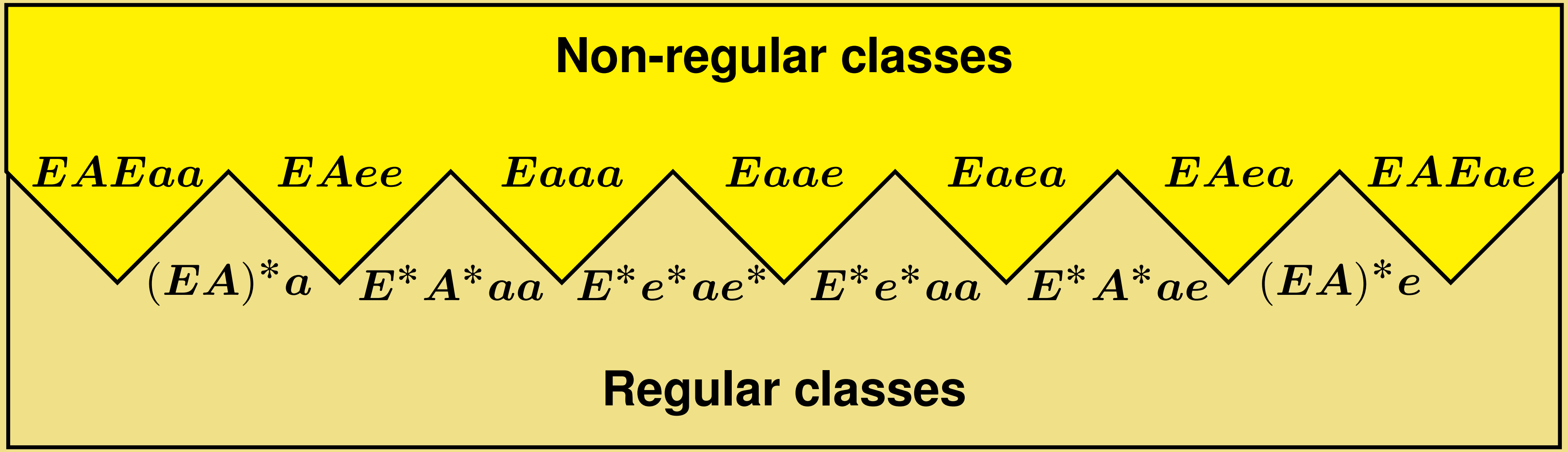


Schüttler 2

If the sentence at hand obeys a formal norm
Just bring it into normal form

Second-Order Logic on Strings: Results

- Second-order logic on strings has been considered in [Eiter, Gottlob, Schwentick 2001]

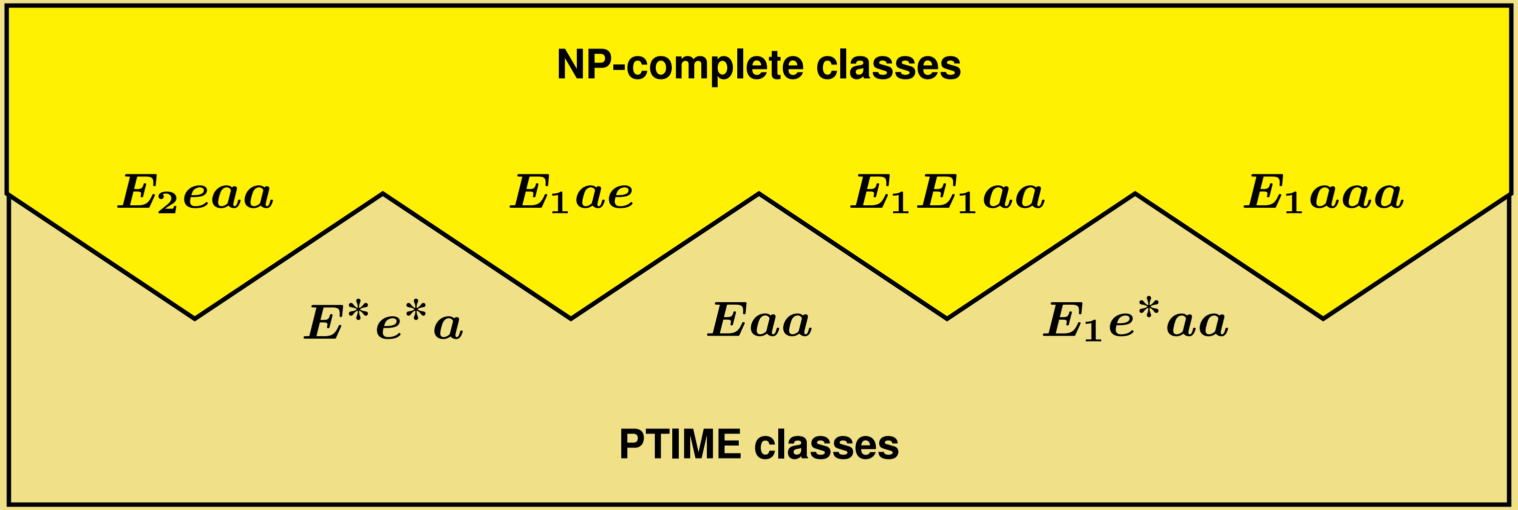


- Here, the boundary between regular and non-regular fragments is determined
- But there is no complete complexity classification of the non-regular fragments

Contents

ESO on Graphs: Results

- The "critical set" S is the same for "arbitrary structures", directed graphs and undirected graphs
- It was determined in [Gottlob, Kolaitis, Schwentick, 2004]
- The following dichotomy was shown:



ESO on Graphs: Upper Bounds

- E^*e^*a: can only express **FO**-properties ☞ not too hard to see

- E_1e^*aa: Reduces to 2-CNF and is thus solvable in **NL**
- In particular: formula $\varphi_1 =$

$$\exists P \exists x_1 \exists x_2 \forall y_1 \forall y_2 \big(P(x_1) \wedge \neg P(x_2) \wedge (P(y_1) \wedge \neg P(y_2) \rightarrow (\neg E(y_1, y_2) \wedge \neg E(y_2, y_1))\big)$$

expresses a **LOGSPACE** problem ☞ DISCONNECTIVITY

- Eaa: Just as in the case of strings guessing binary (or higher arity) relations is not really helpful
- Thus Eaa reduces to E_1aa which in turn is covered by E_1e^*aa

ESO on Graphs: Lower Bounds (1/2)

- E_1E_1aa:
  - Consider φ_2 =

    $$\exists P_1 \exists P_2 \forall y_1 \forall y_2 \Big(E(y_1, y_2) \to \bigwedge_{i=0,1,2} (\neg(\theta_i(y_1) \land \theta_i(y_2)))\Big)$$
  - It expresses that a given undirected graph is 3-colourable

- E_2eaa:
  - Consider φ'_2 =

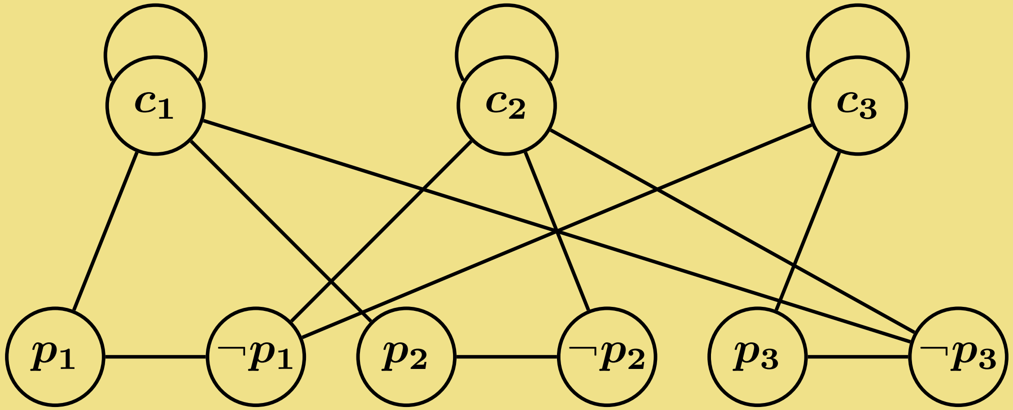
    $$\exists R \exists x \forall y_1 \forall y_2 \Big(E(y_1, y_2) \to \bigwedge_{i=0,1,2} (\neg(\theta'_i(y_1, x) \land \theta'_i(y_2, x)))\Big),$$

    where, e.g., $\theta'_0(z_1, z_2) \stackrel{\text{def}}{=} \neg R(z_1, z_2) \land \neg R(z_2, z_1)$

- E_1aaa: Reduction from POSITIVE-ONE-IN-THREE-SAT

ESO on Graphs: Lower Bounds (2/2)

- E_1ae: Reduction from SAT
  - Encode $(p_1 \vee p_2 \vee \neg p_3) \wedge (\neg p_1 \vee \neg p_2 \vee \neg p_3) \wedge (\neg p_1 \vee p_3)$ by the graph



  - A propositional formula χ is satisfiable if and only if $\varphi_3 = \exists T \forall x \exists y \big((E(x,x) \wedge E(x,y) \wedge \neg E(y,y) \wedge T(y)) \vee (\neg E(x,x) \wedge E(x,y) \wedge \neg E(y,y) \wedge \neg(T(x) \leftrightarrow T(y))\big)$ holds in its graph

- Observation: This reduction seems to rely on the ability to distinguish two kinds of nodes: nodes with and nodes without self-loop

- What changes if we consider only **basic graphs**: undirected graphs **without self-loops**?
  ☞ Name after Tantau

- All proofs for fragments other than E_1ae survive

E^*ae on Basic Graphs (1/3)

- $\varphi_4 = \exists P_1 \exists P_2 \forall x \exists y \big(E(x,y) \wedge$
  $((\theta_0(x) \wedge \theta_1(y)) \vee (\theta_1(x) \wedge \theta_2(y)) \vee (\theta_2(x) \wedge \theta_0(y)))\big)$
  is an E^*ae formula which can use of self-loops ☞ always: $x \neq y$

- φ_4 expresses that each connected component of a graph has a cycle whose length is a multiple of three

- Whether this property can be tested in polynomial time had been open for some time
- In 1988, Thomassen showed that it is indeed in **P** by a very nice argument

- He proved that for every m there is a k such that each graph of tree width $\geqslant k$ has a cycle whose length is a multiple of m
- This yields a nice algorithm:
  - If G has tree width $\geqslant k$, answer "yes"
  - Otherwise check $G \models \varphi_4$ in linear time

☞ thanks to Courcelle's Theorem

Schüttler 3

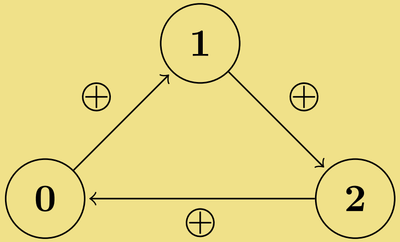
If the structures you deal with resemble a tree
Courcelle has a tool that lets tremble the sea

E^*ae on Basic Graphs (2/3)

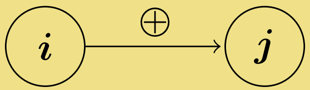
- It turns out that on basic graphs, all E^*ae formulas can be evaluated in **P** ☞[Gottlob, Kolaitis, Schwentick 2004]

- First step in proof: Reduce E^*ae to E_1^*ae
  ☞ similar as for Eaa

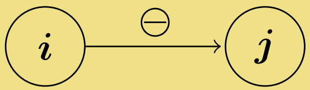
- Second step: Evaluation of E_1^*ae formulas can be reduced to a *Pattern Saturation Problem* for some pattern graph P:



- This problem asks whether the nodes of G can be "coloured" by the colours (vertices) of the pattern such that for each node u of colour i there is a node v of colour j such that
  - u and v are neighbours and occurs in P



  - u and v are non-neighbours and occurs in P



- The proof that the *Pattern Saturation Problem* is in **P** uses tree width as a catalyst similarly as for multiple-of-m-cycles

- The most complicated step is to show (basically) that if G can be saturated by a mixed cycle C of P then G has a small "cycle" that is saturated by C

- Some further important tools for the proof:

- ..

E^*ae on Basic Graphs (3/3)

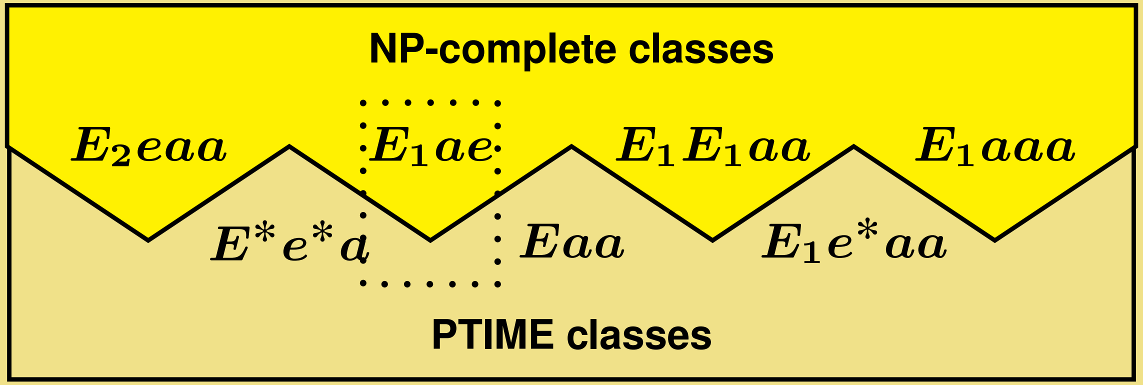
- [Tantau 2014] nailed down the exact complexity of all tractable fragments

- On basic graphs,
  - E^*ae is actually complete for **LOGSPACE**
  - E^*a is also complete for **LOGSPACE**
  - E_1e^*aa is complete for **NL**
  - E_1ae is even in **FO**

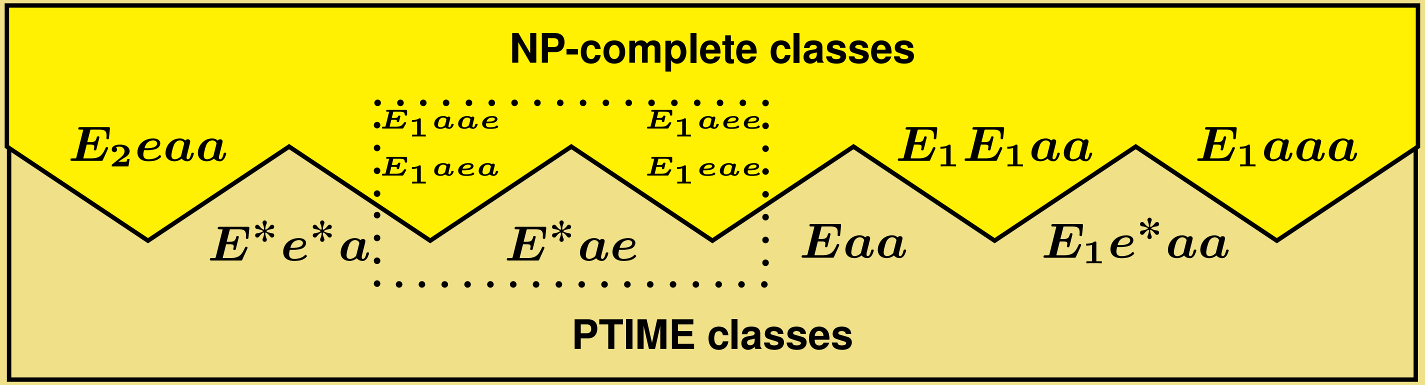
- On directed and undirected graphs
  - E_1e^*aa and E^*a are complete for **NL**

ESO **on Basic Graphs: Summary**

- Graphs:

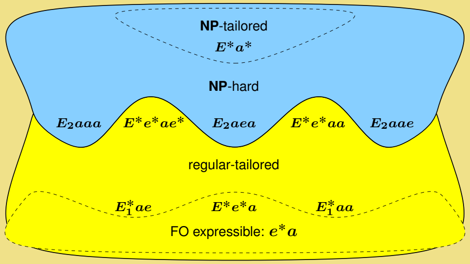
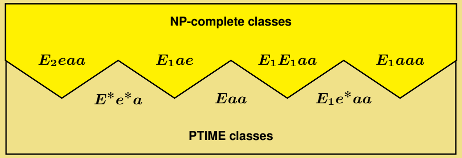
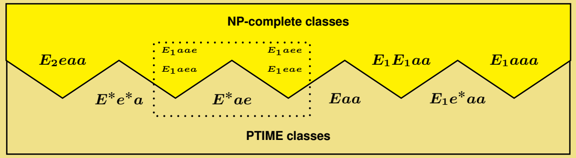
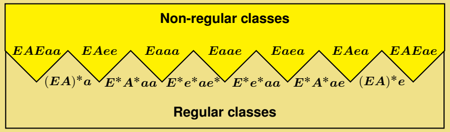


- Basic graphs:



Contents

Summary

	Strings	Graphs	Basic Graphs
ESO			
SO		???	???

- Together with Tantau's work, the classification of quantifier-prefix based classes of ESO is rather complete
- For SO it remains incomplete on strings and to be done on graphs

- Future directions:
  - More fine-grained syntactical analysis
  - Other than quantifier-prefix based fragments
  - Take into account other syntactic concept like separation ☞ Voigt et al.
  - A system?

- Is there anything between strings and graphs?

Bonus Result: ESO on structures with unary functions

- [Barbanchon, Grandjean 2004] studied ESO over structures consisting of unary functions ☞ Think: list structures

- They show that the formula
$$\varphi_6 \stackrel{\text{def}}{=} \exists U \forall x (U(x) \vee U(f(x)) \wedge (\neg U(x) \vee \neg U(f(x)) \vee \neg U(g(x)))$$
over structures with unary functions f and g expresses an **NP**-complete problem

- Furthermore, they prove that this is the *unique minimal* **NP**-complete problem with respect to expressibility in ESO and
  - number of functions
  - number and arity of quantified relations
  - number of first-order variables
  - number of clauses as CNF
  - multiset of CNF-clause sizes (2 & 3)
  - number of clauses as DNF (3)
  - multiset of DNF-clause sizes (2 & 2 & 2)

References

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One more...

Database and Artificial Intelligence Group

Schüttler 4

Isn't this Theorem Shot Lab gorgeous?
The spirit behind it is Gottlob Schorschus

☞[Lindner-Schwentick, Schwentick 2018]

Let us go on

Schüttler 5

After all this fuzz about binary words
Let us go and eat in the winery birds