1	2	3	4	Σ	Grade

6.0/4.0 VU Formale Methoden der Informatik

185.291 **March 15, 2024**

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Block 1.) Recall the following decision problem from the lecture:

> **3-COLORABILITY**(**3-COL**)
>
> INSTANCE: An undirected graph $G = (V, E)$, where V is the set of vertices and E is the set of edges.
>
> QUESTION: Does there exist a total function μ from vertices in V to values in $\{1, 2, 3\}$ such that $\mu(v_1) \neq \mu(v_2)$, for any edge $[v_1, v_2] \in E$?

Consider now the following decision problem:

> **4-COLORABILITY**(**4-COL**)
>
> INSTANCE: An undirected graph $G = (V, E)$, where V is the set of vertices and E is the set of edges.
>
> QUESTION: Does there exist a total function μ from vertices in V to values in $\{1, 2, 3, 4\}$ such that $\mu(v_1) \neq \mu(v_2)$, for any edge $[v_1, v_2] \in E$?

1.a) The following describes a reduction from **3-COL** to **4-COL**. Given an arbitrary instance $G = (V, E)$ of **3-COL**, let G' be the following graph:

$$G' = (V \cup \{u\}, E \cup \{[v, u] \mid v \in V\}).$$

Show that G is a yes-instance of **3-COL** if and only if G' is a yes-instance of **4-COL**.

(9 points)

1.b) It is known that **3-COL** is NP-complete. With this in mind, please answer the following questions and **explain your answers**:

- Is **4-COL** NP-hard?
- Is **4-COL** NP-complete? If so, provide a certificate relation and argue that it is polynomially balanced and polynomially decidable.

(6 points)

Block 2.)

2.a) Show that $b[j] \doteq f \rightarrow b\langle j \triangleleft f\rangle \doteq b$ is $\mathcal{T}_A^{=}$-valid.

Besides the equality axioms, you have the following ones for the arrays.

$\forall a\, \forall i\, \forall j\, \big(i \doteq j \rightarrow a[i] \doteq a[j]\big)$	(array congruence)
$\forall a\, \forall v\, \forall i\, \forall j\, \big(i \doteq j \rightarrow a\langle i \triangleleft v\rangle[j] \doteq v\big)$	(read-over-write 1)
$\forall a\, \forall v\, \forall i\, \forall j\, \big(i \not\doteq j \rightarrow a\langle i \triangleleft v\rangle[j] \doteq a[j]\big)$	(read-over-write 2)
$\forall a\, \forall b\, \big((\forall j\; a[j] \doteq b[j]) \leftrightarrow a \doteq b\big)$	(extensionality)

Please be precise and justify every proof step.

(12 points)

2.b) Consider the following clause set $\hat{\delta}(\varphi)$ which has been derived from an (unknown) formula φ by an improved version of Tseitin's translation (atoms have not been labeled).

$$\begin{array}{llllll}
C_1: & \ell_1 \vee \neg x \vee \neg y & C_2: & \neg \ell_1 \vee x & C_3: & \neg \ell_1 \vee y \\
C_4: & \neg \ell_2 \vee \neg y \vee z & C_5: & \ell_2 \vee y & C_6: & \ell_2 \vee \neg z \\
C_7: & \neg \ell_3 \vee \neg \ell_1 \vee z & C_8: & \ell_3 \vee \ell_1 & C_9: & \ell_3 \vee \neg z \\
C_{10}: & \neg \ell_4 \vee \neg x \vee \ell_2 & C_{11}: & \ell_4 \vee x & C_{12}: & \ell_4 \vee \neg \ell_2 \\
C_{13}: & \neg \ell_5 \vee \neg \ell_3 \vee \ell_4 & C_{14}: & \neg \ell_5 \vee \ell_3 \vee \neg \ell_4 & C_{15}: & \ell_5 \vee \ell_3 \vee \ell_4 \\
C_{16}: & \ell_5 \vee \neg \ell_3 \vee \neg \ell_4 & & & &
\end{array}$$

(i) Reconstruct φ from $\hat{\delta}(\varphi)$ with a minimal number of connectives.

(ii) Suppose you want to check the validity of φ by resolution and $\hat{\delta}(\varphi)$. How do you proceed ? Please explain your approach!

(3 points)

Block 3.)

3.a) Let p be the following IMP program loop, containing the integer-valued program variables x, y, z:

$$
\begin{array}{l}
\textbf{while } y \neq z \textbf{ do} \\
\quad x := x + 2; \\
\quad z := x - 2; \\
\quad y := y + z \\
\textbf{od}
\end{array}
$$

Which of the following program assertions are inductive loop invariants of p?

- $I_1 : \quad x - z = 2$
- $I_2 : \quad y = 1 \land z = 1$
- $I_3 : \quad y = x + z$

Give formal details justifying your answer. That is, if an assertion is an inductive loop invariant, provide a formal proof of it based on Hoare logic or using weakest liberal preconditions. If an assertion is not an inductive loop invariant, give a counterexample and justify your answer.

(9 points)

3.b) Let x, y be integer-valued variables and p an arbitrary IMP program.

Assume that the Hoare triple $[x \neq y] \;\; p \;\; [x = y]$ for total correctness is valid. Then, which of the Hoare triples below are valid:

(i) $[x = 1 \wedge y = 1]$ **if** $x \neq y$ **then** p **else** **abort** $[x = y]$?

(ii) $[x = 1 \wedge y = 2]$ **if** $x \neq y$ **then** p **else** **abort** $[x = y]$?

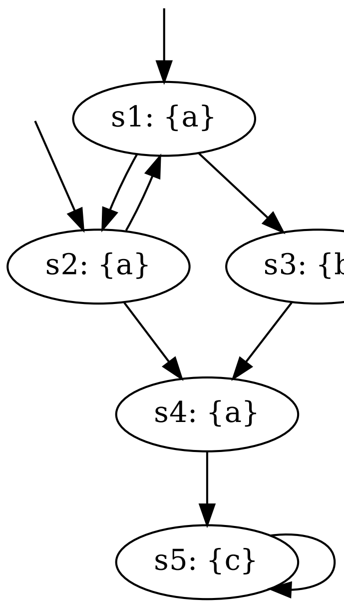
For each of the triples, if the triple is valid, justify your answer. Otherwise, provide a counterexample.

(6 points)

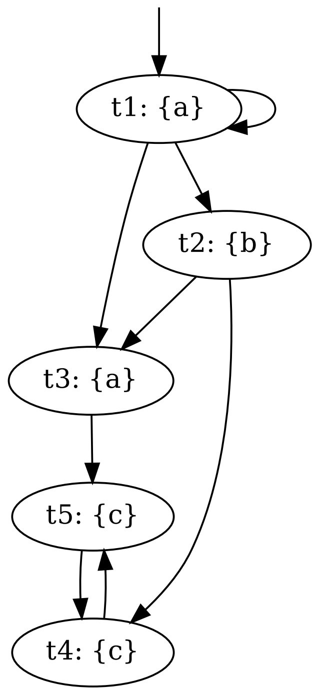
Block 4.)

4.a) Consider the Kripke structures M_1 and M_2. The initial states of M_1 are s_1 and s_2 and the initial state of M_2 is t_1.

Kripke structure M_1:



Kripke structure M_2:

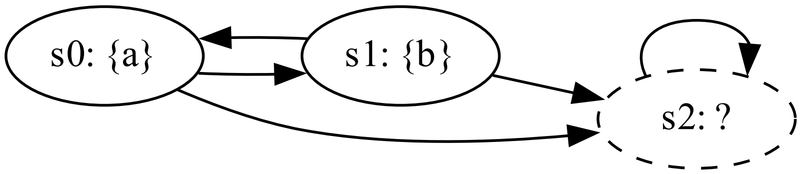


i. Check whether M_2 simulates M_1, i.e., provide a simulation relation that witnesses $M_1 \preceq M_2$, or briefly explain why M_2 does not simulate M_1.

ii. Check whether M_1 simulates M_2, i.e., provide a simulation relation that witnesses $M_2 \preceq M_1$, or briefly explain why M_1 does not simulate M_2.

(4 points)

4.b) Consider the following Kripke structure M:



For each of the following formulae φ,

i. indicate whether the formula is in CTL, LTL, and/or CTL*, and
ii. indicate for which combinations of variables (if any) in $s2$, the formula is true in state $s0$, i.e., for which Kripke structures M do we have $M, s0 \models \varphi$? (Note: if φ is a path formula, we mean $M, s0 \models \mathbf{A}\varphi$.)

φ	CTL	LTL	CTL*	$s2 : \{a\}$	$s2 : \{b\}$	$s2 : \{a, b\}$
$\mathbf{F}\, b$	☐	☐	☐	☐	☐	☐
$\mathbf{E}\ (b\ \mathbf{U}\ a)$	☐	☐	☐	☐	☐	☐
$\mathbf{XX}\neg b$	☐	☐	☐	☐	☐	☐
$\mathbf{G}\,\neg(a \leftrightarrow b)$	☐	☐	☐	☐	☐	☐
$\mathbf{EX}(b \wedge \mathbf{G}a)$	☐	☐	☐	☐	☐	☐

(5 points)

4.c) An LTL formula is a *tautology* if it holds for every Kripke structure M and every path π in M. For each of the following formulas, prove that it is a tautology, or find a Kripke structure M and path π in M for which the formula does not hold and justify your answer.

i. $\mathbf{G}(\mathbf{F}a \leftrightarrow \mathbf{F}b) \rightarrow (\mathbf{GF}a \wedge \mathbf{GF}b)$

ii. $(\mathbf{GF}a \wedge \mathbf{GF}b) \rightarrow \mathbf{G}(\mathbf{F}a \leftrightarrow \mathbf{F}b)$

(6 points)

Grading scheme: 0–29 nicht genügend, 30–35 genügend, 36–41 befriedigend, 42–47 gut, 48–60 sehr gut