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6.0/4.0 VU Formale Methoden der Informatik (185.291)

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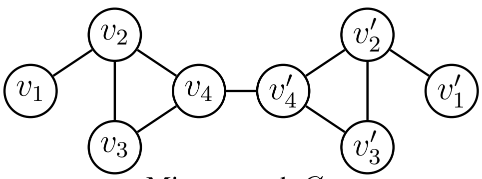
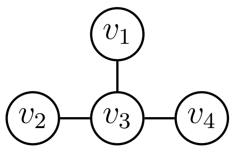
Block 1.) We call an undirected graph $G = (V, E)$ a *mirror-graph* if the following holds:

1. $V = V_1 \cup V_2$ with $|V_1| = |V_2|$, $V_1 \cap V_2 = \emptyset$,
2. $E = E_1 \cup E_2 \cup \{(v_1^*, v_2^*)\}$ with $E_i \subseteq V_i \times V_i$, $v_i^* \in V_i$ for $i \in \{1, 2\}$,
3. there is a bijective mapping $m : V_1 \to V_2$ s.t. $(v_i, v_j) \in E_1$ if and only if $(m(v_i), m(v_j)) \in E_2$,
4. For the two vertices v_1^*, v_2^* from 2. it holds $m(v_1^*) = v_2^*$.

In other words, a mirror graph consists of two "mirrored" copies of a graph that are connected by one "bridge"-edge between two corresponding vertices.
For example:
G_1 is a mirror graph with $m(v_i) = v_i'$ for $1 \leq i \leq 4$.
G_2 is not a mirror graph (none of the three edges could be the "bridge"-edge).



Mirror-graph G_1



Non-mirror-graph G_2

Consider the following variant of the problem **3-COLORABILITY**.

> **MIRROR-GRAPH 3-COLORABILITY (M-3COL)**
>
> INSTANCE: A mirror-graph $G = (V, E)$.
>
> QUESTION: Does there exist a function μ from vertices in V to values in $\{0, 1, 2\}$ such that $\mu(v_1) \neq \mu(v_2)$ for any edge $(v_1, v_2) \in E$?

1.a) Briefly argue why **M-3COL** is in NP. You may use the fact that **3-COLORABILITY** is NP-complete. **(2 points)**

1.b) The following function f provides a polynomial-time many-one reduction from the problem **3-COLORABILITY** to **M-3COL**: for an undirected graph $G = (V, E)$, let $f(G) = (V', E')$, where

$$
\begin{aligned}
V' &= V \cup \{v' \mid v \in V\}; \text{ and} \\
E' &= E \cup \{(v_i', v_j') \mid (v_i, v_j) \in E\} \cup \{(v_b, v_b')\},
\end{aligned}
$$

where $v_b \in V$ is a arbitrary vertex from G.

Show the correctness of the reduction, i.e., show:
G is a yes-instance of **3-COLORABILITY** $\iff f(G)$ is a yes-instance of **M-3COL**.

(9 points)

1.c) In what follows assume the reduction from **3-COLORABILITY** to **M-3COL** is correct, and recall that **3-COLORABILITY** is NP-complete.

Tick the correct statements (for ticking a correct statement a certain number of points is given; ticking an incorrect statement results in a substraction of the same amount; you cannot go below 0 points):

- ○ **M-3COL** is NP-hard
- ○ An exponential-time algorithm for **M-3COL** shows P=NP
- ○ There is a polynomial-time many-one reduction from **SAT** to **M-3COL**

(4 points)

Block 2.)

2.a) Let $\mathcal{T}_E^{di}$ be a first-order theory containing all axioms of the theory of equality $\mathcal{T}_E$ (including substitution axiom schemes for p and f) and the two axioms:

$$\forall x \forall y\ \Big(p(x,y) \rightarrow \big(p(x, f(x,y)) \wedge p(f(x,y), y)\big)\Big) \qquad \text{(p-density)}$$

$$\forall x \forall y\ \Big(p(x,y) \rightarrow x \neq y\Big) \qquad \text{(p-irreflexivity)}$$

Use the semantic argument method to prove the following:
Let $\mathcal{I}$ be a $\mathcal{T}_E^{di}$-interpretation structure with $\mathcal{I} \models p(a,b)$, then it holds that $\mathcal{I} \models f(a,b) \neq a \wedge f(a,b) \neq b \wedge a \neq b$. **(7 points)**

2.b) Let φ be any propositional formula in negation normal form (NNF).

Prove by induction: If φ contains only pure literals, then φ is satisfiable.

Recall that a literal ℓ is pure in a formula φ in NNF, if ℓ occurs in φ, but the complement of ℓ, ℓ^c, does not occur in φ, where ℓ^c is b if ℓ is $\neg b$ and ℓ^c is $\neg b$ if ℓ is b. Please be precise. Indicate which induction concept you use and which statement you prove. **(8 points)**

Block 3.)

3.a) Let p be the following IMP program, containing the integer-valued program variables x, y, z:

$$
\begin{array}{l}
x := 0; y := 10; z := 6; \\
\textbf{while } z > 0 \textbf{ do} \\
\quad y := y - 2; \\
\quad x := x + 2 * y - 4 * z + 5; \\
\quad z := z - 1; \\
\textbf{od}
\end{array}
$$

Give a loop invariant and variant for the **while** loop in p and prove the validity of the total correctness triple $[2z = y + 2 \wedge z \geq 0]\ p\ [y = x + 16]$.

Note: Make sure that your invariant expresses equalities among x, y, z as well equalities among y, z.

(10 points)

3.b) Let p be the following IMP program, containing the integer-valued program variables x, y:

$$\textbf{if } x \neq y \textbf{ then } x := x + 2 \textbf{ else } y := y - 2$$

Which of the following Hoare triples are correct? For each triple, provide a short justification for your answer.

(i) $\{x > 0\}\ p\ \{x > 0\}$

(ii) $\{x < 0\}\ p\ \{x \geq 0\}$

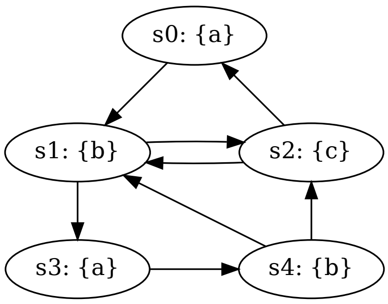
(iii) $\{x = y\}\ p\ \{x < y\}$

(5 points)

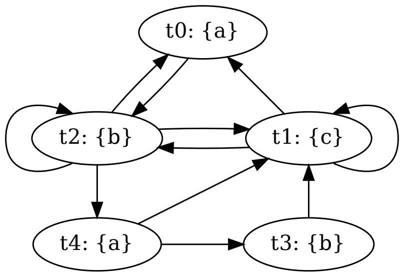
Block 4.)

4.a) Consider the Kripke structures M_1 and M_2. The initial state of M_1 is s_0, the initial state of M_2 is t_0.

Kripke structure M_1:



Kripke structure M_2:



i. Show that M_2 simulates M_1 by providing a non-empty simulation relation that witnesses $M_1 \preceq M_2$.

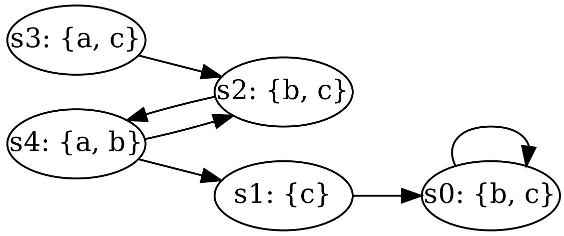
ii. Show that M_1 does **not** simulate M_2 by providing an ACTL* or LTL formula φ that **holds** in the **left** structure, but does **not hold** in the **right** structure, i.e., provide an ACTL* or LTL formula φ such that $M_1 \models \varphi$ but $M_2 \not\models \varphi$.

Hint 1: ACTL* is the universal fragment of CTL*.

(5 points)

4.b) CTL Marking Algorithm

Consider the following Kripke structure M:



Execute the **CTL Marking Algorithm** to determine which states satisfy the following formulae:

i. $\mathbf{E}[a\ \mathbf{U}\ \neg b]$
ii. $\mathbf{AXEG}c$

Use the answer templates below.

In particular,

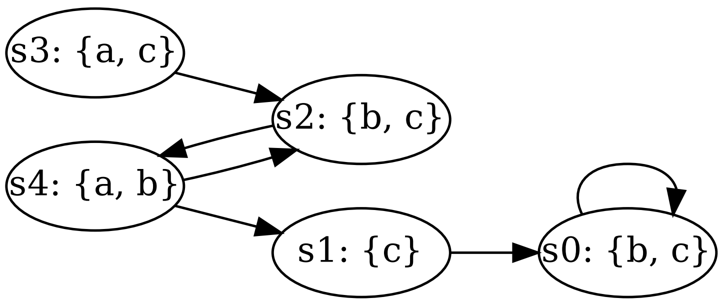
i. Transform Φ into an equivalent formula Φ' that uses temporal operators **EX**, **EU** and **EG**.
ii. List the subformulae of Φ'.
iii. For increasing nesting depth, iteratively determine the states marked by subformulae $\phi_0, \psi_0, \phi_1, \psi_1, \ldots$ of Φ'.
iv. Finally, give the return value of the Marking Algorithm. That is, list the states s_i that satisfy formula Φ, i.e., for which states do we have that $M, s_i \models \Phi$?

Hint: Recall that the algorithm starts by marking propositional atoms ϕ_0. It then iteratively marks boolean combinations ψ_i of subformulas ϕ_i, and temporal operator applications $\phi_{i+1} = \circ\ \psi_i$ where $\circ \in \{\mathbf{EX}, \mathbf{EU}, \mathbf{EG}\}$.

i) Answer template for $\Phi = \mathbf{E}[a\ \mathbf{U}\ \neg b]$

Subformulae of Φ: ______________________________

Annotate the states of M with the subformulae by which the Marking Algorithm marks them:



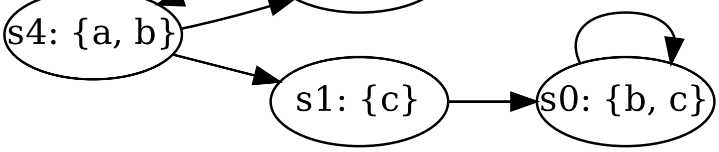
States satisfying Φ: ______________________________

ii) Answer template for $\Phi = \mathbf{AXEG}c$

Equivalent formula $\Phi' \equiv \Phi$ using only **EX**, **EU**, **EG**: ____________________

Subformulae of Φ': ____

Annotate the states of M with the subformulae by which the Marking Algorithm marks them:



States satisfying Φ: ____

(6 points)

4.c) LTL Tautologies

An LTL formula is a *tautology* if it holds for every Kripke structure M and every path π in M. For each of the following formulas, prove that it is a tautology, or find a Kripke structure M and path π in M for which the formula does not hold and justify your answer.

i. $(a \vee b)\ \mathbf{U}\ (a \wedge b) \Rightarrow (\mathbf{G}a \wedge \mathbf{F}b)$

ii. $(\mathbf{G}a \wedge \mathbf{F}b) \Rightarrow (a \vee b)\ \mathbf{U}\ (a \wedge b)$

(4 points)

Grading scheme: 0–29 nicht genügend, 30–35 genügend, 36–41 befriedigend, 42–47 gut, 48–60 sehr gut