1.) An undirected graph \((V, E)\) is called a *degree-restricted graph* if for each vertex \(v \in V\) it holds that the degree of \(v\) is either 1 or greater than 3.

Examples:
- \(((a, b, c, d, e), \{(a, b), (a, c), (a, d), (a, e)\})\) is degree-restricted since vertex \(a\) has degree 4 and vertices \(\{b, c, d, e\}\) have all degree 1.
- \(((a, b, c, d), \{(a, b), (b, c), \ldots\})\) is not degree-restricted since vertices \(b, c\) have degree 2.
- The complete graph \(K_4\), i.e. a clique with 4 vertices, is also not degree-restricted since all vertices have degree 3.

Consider the following variant of the 3-coloring problem:

**3-COLORABILITY-DEGREE-RESTRICTED (3COLD)**

**INSTANCE:** A degree-restricted graph \(G = (V, E)\).

**QUESTION:** Does there exist a function \(\mu\) from vertices in \(V\) to values in \(\{0, 1, 2\}\) such that \(\mu(v_1) \neq \mu(v_2)\) for any edge \([v_1, v_2] \in E\).

(a) The following function \(f\) provides a polynomial-time many-one reduction from **3COL** to **3COLD:** for an undirected graph \(G = (\{v_1, \ldots, v_n\}, E)\), add for each vertex \(v_i\)
- four new vertices \(x_{i1}, x_{i2}, x_{i3}, x_{i4}\), and
- edges \([v_i, x_{i1}], [v_i, x_{i2}], [v_i, x_{i3}], [v_i, x_{i4}]\)

to \(G\), and let \(f(G)\) be the resulting graph. Note that \(f(G)\) is indeed degree-restricted, since the new vertices \(x_{i1}\) all have degree 1 and vertices \(v_i\) have at least degree 4.

Show that \(G\) is a yes-instance of **3COL** \iff \(f(G)\) is a yes-instance of **3COLD**.

(b) Let us assume the reduction from **3COL** to **3COLD** is correct. Argue briefly why we can then conclude that **3COLD** is NP-complete.
2.) We consider the function $P$ which was introduced by Rózsa Péter in 1935.

**Input:** $x, y$, two non-negative integers  
**Output:** The computed integer value for $x, y$

if $x == 0$ then  
    return $2y + 1$;  
else if $y == 0$ then  
    return $P(x - 1, 1)$;  
else return $P(x - 1, P(x, y - 1))$;  

**Algorithm 1:** The function $P$

(a) Let $\mathbb{N}_0$ denote the set of natural numbers including $0$. Use well-founded induction to show 
$$\forall x \forall y \left( (x \in \mathbb{N}_0 \land y \in \mathbb{N}_0) \rightarrow P(x, y) > 2x + 2y \right).$$

(12 points)

(b) Consider the clauses $C_1, \ldots, C_6$ in **dimacs** format (in this order from top to bottom, shown in the box) which are given as input to a SAT solver.

- Apply CDCL using the convention that if a variable is assigned as a decision, then it is assigned 'true'. Select variables as decisions in increasing order of their respective integer IDs in the **dimacs** format, starting with variable 1.
- When the first conflict occurs, draw the complete implication graph, mark the first UIP, give the derivation of the learned asserting clause that corresponds to the first UIP, and stop CDCL. You do not have to solve the formula!

(3 points)

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3.) (a) Let $p$ be the following IMP program loop, containing the integer-valued program variables $i, x, y$:

\[
\begin{align*}
x &:= 10; \\
y &:= 10; \\
\textbf{while } &i \geq 0 \textbf{ do} \\
x &:= x - 4 \times i; \\
y &:= y + 4 \times i; \\
i &:= i - 1; \\
\textbf{od}
\end{align*}
\]

Give a loop invariant and variant for the \textbf{while} loop in $p$ and prove the validity of the total correctness triple $[i > 0] \ p \ [x + y = 20]$.

(10 points)

(b) Let $p$ be the following IMP program loop, containing the integer-valued program variables $i$ and $x$:

\[
\begin{align*}
\textbf{while } &i \neq 3 \textbf{ do} \\
x &:= 3 \times x; \\
i &:= i - 1; \\
\textbf{od}
\end{align*}
\]

Provide a non-trivial pre-condition $A$, such that:

(i) the total correctness triple $[A] \ p \ [x \geq 27]$ is valid;

(ii) the partial correctness triple $\{A\} \ p \ \{x \geq 27\}$ is valid, but the total correctness triple $[A] \ p \ [x \geq 27]$ is not valid.

Trivial means equivalent to true or false, so your precondition $A$ should not be equivalent to true or false. Justify your answer!

(5 points)
4.) (a) Consider the Kripke structures $M_1$ and $M_2$. The initial state of $M_1$ is $s_0$, the initial state of $M_2$ is $t_0$.

**Kripke structure $M_1$:**

- $s_0$: \{c\}
- $s_1$: \{b\}
- $s_2$: \{a\}
- $s_3$: \{a\}
- $s_4$: \{b\}

**Kripke structure $M_2$:**

- $t_0$: \{c\}
- $t_1$: \{a\}
- $t_2$: \{c\}
- $t_3$: \{b\}
- $t_4$: \{b\}

i. Check whether $M_2$ simulates $M_1$, i.e., provide a simulation relation that witnesses $M_1 \preceq M_2$, or explain why $M_2$ does not simulate $M_1$.

ii. Check whether $M_1$ simulates $M_2$, i.e., provide a simulation relation that witnesses $M_2 \preceq M_1$, or explain why $M_1$ does not simulate $M_2$.

(4 points)
(b) Consider the following Kripke structure $M$:

For each of the following formulae $\varphi$,

i. indicate whether the formula is in CTL, LTL, and/or CTL*, and

ii. list the states $s_i$ on which the formula $\varphi$ holds; i.e. for which states $s_i$ do we have $M, s_i \models \varphi$?

(If $\varphi$ is a path formula, list the states $s_i$ such that $M, s_i \models A \varphi$.)

<table>
<thead>
<tr>
<th>$\varphi$</th>
<th>CTL</th>
<th>LTL</th>
<th>CTL*</th>
<th>States $s_i$</th>
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<tr>
<td>$G(b \land c)$</td>
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<td>$EFGa$</td>
<td>☐</td>
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<tr>
<td>$E(Gc \land GFb)$</td>
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<td>☐</td>
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<tr>
<td>$F(a \land Xc)$</td>
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</tr>
<tr>
<td>$E(\neg a \ U c)$</td>
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</table>

(5 points)

(c) **LTL tautologies**

An LTL formula is a tautology if it holds for every Kripke structure $M$ and every path $\pi$ in $M$. For each of the following formulas, prove that it is a tautology, or find a Kripke structure $M$ and path $\pi$ in $M$ for which the formula does not hold and justify your answer.

i. $\neg(a \ U b) \Rightarrow (G\neg b \lor F\neg a)$

ii. $(G\neg b \lor F\neg a) \Rightarrow \neg(a \ U b)$

(6 points)