1	2	3	4	Σ

6.0/4.0 VU Formale Methoden der Informatik (185.291)

Dec 10, 2021

Kennz. (study id)	Matrikelnummer (student id)	Nachname (surname)	Vorname (first name)

1.) An undirected graph (V, E) is called a *degree-restricted graph* if for each vertex $v \in V$ it holds that the degree of v is 1 or even (i.e. 1, 2, 4, 6, ...).

Examples: $(\{a, b, c, d\}, \{[a, b], [b, c], [c, d]\})$ is degree-restricted since vertices a and d have degree 1, and b and c have an even degree 2. $(\{a, b, c, d\}, \{[a, b], [a, c], [a, d]\})$ is not degree-restricted since vertex a has degree 3.

Consider the following variant of the 3-coloring problem:

3-COLORABILITY-DEGREE-RESTRICTED (3COLD)

INSTANCE: A degree-restricted graph $G = (V, E)$.

QUESTION: Does there exist a function μ from vertices in V to values in $\{0, 1, 2\}$ such that $\mu(v_1) \neq \mu(v_2)$ for any edge $[v_1, v_2] \in E$.

(a) The following function f provides a polynomial-time many-one reduction from **3COL** to **3COLD**: for an undirected graph $G = (\{v_1, \dots, v_n\}, E)$, add for each vertex v_i with odd degree > 2

- a new vertex x_i
- an edge $[v_i, x_i]$

to G, and let $f(G)$ be the resulting degree-restricted graph.
Show that G is a yes-instance of **3COL** $\Longleftrightarrow$ $f(G)$ is a yes-instance of **3COLD**.

(9 points)

(b) In what follows assume the reduction from **3COL** to **3COLD** is correct, and recall that **3COL** is NP-complete.
Tick the correct statements (for ticking a correct statement a certain number of points is given; ticking an incorrect statement results in a substraction of the same amount; you cannot go below 0 points):

- Since **3COL** is NP-complete, our reduction shows that **3COLD** is NP-hard.
- Since **3COL** is NP-hard, our reduction shows that **3COLD** is NP-hard.
- Since **3COL** is in NP, our reduction shows that **3COLD** is NP-hard.
- Since **3COLD** is a special case of **3COL**, it follows that **3COLD** is contained in NP.
- Since **3COLD** is a special case of **3COL**, it follows that **3COLD** is NP-hard.
- Since **3COLD** is a special case of **3COL**, it follows that **3COLD** is NP-complete (even without the above reduction).

(6 points)

2.) (a) Consider the following clause set $\hat{\delta}(\varphi)$ which has been derived from an (unknown) formula φ by Tseitin's translation (atoms have not been labeled).

$$\begin{array}{llllll}
C_1: & \ell_1 \vee \neg x \vee \neg y & C_2: & \neg \ell_1 \vee x & C_3: & \neg \ell_1 \vee y \\
C_4: & \neg \ell_2 \vee \neg y \vee z & C_5: & \ell_2 \vee y & C_6: & \ell_2 \vee \neg z \\
C_7: & \neg \ell_3 \vee \neg \ell_1 \vee z & C_8: & \ell_3 \vee \ell_1 & C_9: & \ell_3 \vee \neg z \\
C_{10}: & \neg \ell_4 \vee \neg x \vee \ell_2 & C_{11}: & \ell_4 \vee x & C_{12}: & \ell_4 \vee \neg \ell_2 \\
C_{13}: & \neg \ell_5 \vee \neg \ell_3 \vee \ell_4 & C_{14}: & \ell_5 \vee \ell_3 & C_{15}: & \ell_5 \vee \neg \ell_4
\end{array}$$

(i) Reconstruct φ from $\hat{\delta}(\varphi)$.

(ii) Prove the correctness of the propositional resolution rule.

(iii) Prove the validity of φ by resolution (no additional translation to normal form is allowed!). You are allowed to add a single unit clause (i.e., a clause containing exactly one literal). Please explain your approach!

(7 points)

(b) Clarify the logical status of each of the following formulas. If one is $\mathcal{T}_{cons}^{E}$-valid or $\mathcal{T}_{cons}^{E}$-unsatisfiable, then prove it using the semantic argument method. If one is $\mathcal{T}_{cons}^{E}$-satisfiable but not $\mathcal{T}_{cons}^{E}$-valid, then present a satisfying and a falsifying interpretation. Argue formally that the formula evaluates to true resp. false under the constructed interpretations.

$$\varphi_0 \colon \neg atom(x) \wedge car(x) \doteq y \wedge cdr(x) \doteq z \wedge x \not\doteq cons(y, z)$$
$$\begin{aligned} \varphi_1 \colon\ & cons(car(x), cdr(x)) \doteq cons(y, z) \wedge cons(car(x), cdr(x)) \not\doteq x \\ & \rightarrow x \not\doteq cons(y, z) \end{aligned}$$

Besides the equality axioms, the following axioms of $\mathcal{T}_{cons}^{E}$ may be helpful.

- $\forall x, y\, car(cons(x, y)) \doteq x$ (left projection)
- $\forall x, y\, cdr(cons(x, y)) \doteq y$ (right projection)
- $\forall x\, \neg atom(x) \rightarrow cons(car(x), cdr(x)) \doteq x$ (construction)
- $\forall x, y\, \neg atom(cons(x, y))$ (atom)

(8 points)

(15 points)

3.) (a) Let p be the following IMP program loop, containing the integer-valued program variables x, i, b:

$$\begin{array}{l} \textbf{while } x < b \textbf{ do} \\ \quad x := i * i; \\ \quad i := i + 1; \\ \textbf{od} \end{array}$$

Which of the following program assertions are inductive loop invariants of p?

- $I_1 : \quad x - i * i = 0$
- $I_2 : \quad true$
- $I_3 : \quad x \leq b$

Give formal details justifying your answer. That is, if an assertion is an inductive loop invariant, provide a formal proof of it based on Hoare logic. If an assertion is not an inductive loop invariant, give a counterexample.

Note: You need to use the definition of an assertion being inductive invariant.

(6 points)

(b) Let p be the following IMP program loop, containing the integer-valued program variables i, x, y:

$$\begin{array}{l} \textbf{while } i < 10 \textbf{ do} \\ \quad i := i + 1; \\ \quad x := x - 2; \\ \quad y := y + x + 2 * i; \\ \textbf{od} \end{array}$$

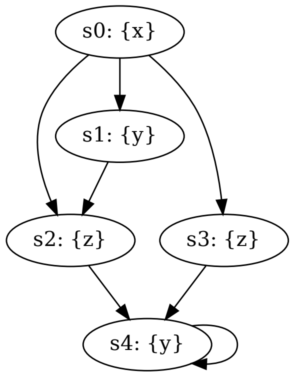
Give a loop invariant for the **while** loop in p and prove the validity of the partial correctness triple $\{i = 0 \land x = 10 \land y = 0\}\ p\ \{y = 100\}$.

Note: Make sure that your invariant expresses equalities among i, x, y as well equalities among i, x.

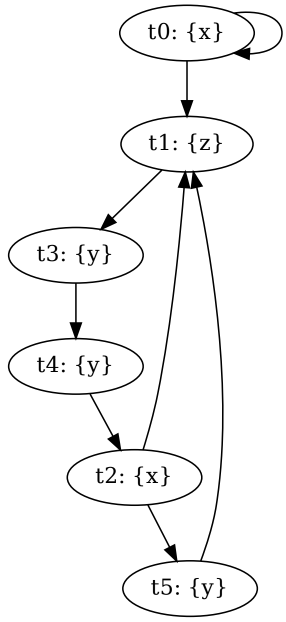
(9 points)

4.) (a) Consider the Kripke structures M_1 and M_2. The initial state of M_1 is s_0, the initial state of M_2 is t_0.

Kripke structure M_1**:**



Kripke structure M_2**:**

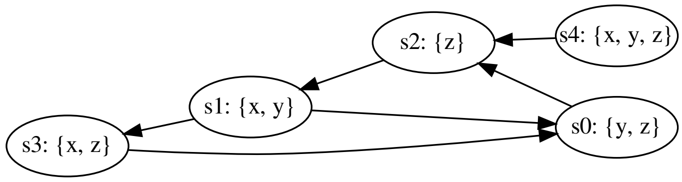


i. Briefly explain why M_2 does not simulate M_1.

ii. Add a minimal set of edges to M_2 such that the extended Kripke structure M_2' simulates M_1. Provide the additional edges and a non-empty simulation relation H that witnesses $M_1 \preceq M_2'$.

(4 points)

(b) Consider the following Kripke structure M:



For each of the following formulae φ,

i. indicate whether the formula is in CTL, LTL, and/or CTL*, and

ii. list the states s_i on which the formula φ holds; i.e. for which states s_i do we have $M, s_i \models \varphi$?
(If φ is a path formula, list the states s_i such that $M, s_i \models \mathbf{A}\varphi$.)

φ	CTL	LTL	CTL*	States s_i
$\mathbf{AF}(x \wedge z)$	☐	☐	☐	
$\mathbf{G}(z)$	☐	☐	☐	
$x\ \mathbf{U}\ y$	☐	☐	☐	
$\mathbf{EF}(x)$	☐	☐	☐	
$\mathbf{AX}(x)$	☐	☐	☐	

(5 points)

(c) **LTL tautologies**

Prove that the following formulas are tautologies, i.e., they hold for every Kripke structure M and every path π in M, or find a Kripke structure M and path π in M, for which the formula does not hold and justify your answer.

i. $a \ \mathbf{U} \ (b \vee \neg a) \Rightarrow \neg \mathbf{G} a \wedge \mathbf{F} b$

ii. $\neg \mathbf{G} a \wedge \mathbf{F} b \Rightarrow a \ \mathbf{U} \ (b \vee \neg a)$

(6 points)

Grading scheme: 0–29 nicht genügend, 30–35 genügend, 36–41 befriedigend, 42–47 gut, 48–60 sehr gut