1.) An undirected graph \((V,E)\) is called a *degree-restricted graph* if for each vertex \(v \in V\) it holds that the degree of \(v\) is odd (i.e. 1, 3, 5,...).

Examples: \((\{a,b,c,d\}, \{[a,b], [a,c], [a,d]\})\) is degree-restricted since vertex \(a\) has degree 3 and vertices \(\{b,c,d\}\) have all degree 1.

\((\{a,b,c,d\}, \{[a,b], [b,c], [c,d]\})\) is not degree-restricted since vertices \(b\) and \(c\) have an even degree 2.

Consider the following variant of the 3-coloring problem:

**3-COLORABILITY-DEGREE-RESTRICTED (3COLD)**

**INSTANCE:** A degree-restricted graph \(G = (V,E)\).

**QUESTION:** Does there exist a function \(\mu\) from vertices in \(V\) to values in \(\{0, 1, 2\}\) such that \(\mu(v_1) \neq \mu(v_2)\) for any edge \([v_1, v_2] \in E\).

(a) The following function \(f\) provides a polynomial-time many-one reduction from 3COL to 3COLD: for an undirected graph \(G = ([v_1, \ldots, v_n], E)\), add for each vertex \(v_i\) with even degree

- a new vertex \(x_i\)
- an edge \([v_i, x_i]\)

to \(G\), and let \(f(G)\) be the resulting degree-restricted graph.

Show that \(G\) is a yes-instance of 3COL \(\iff\) \(f(G)\) is a yes-instance of 3COLD.

(9 points)

(b) In what follows assume the reduction from 3COL to 3COLD is correct, and recall that 3COL is NP-complete.

Tick the correct statements (for ticking a correct statement a certain number of points is given; ticking an incorrect statement results in a subraction of the same amount; you cannot go below 0 points):

- Since 3COL is NP-complete, our reduction shows that 3COLD is NP-hard.
- Since 3COL is NP-hard, our reduction shows that 3COLD is NP-hard.
- Since 3COL is in NP, our reduction shows that 3COLD is NP-hard.
- Since 3COLD is a special case of 3COL, it follows that 3COLD is contained in NP.
- Since 3COLD is a special case of 3COL, it follows that 3COLD is NP-hard.
- Since 3COLD is a special case of 3COL, it follows that 3COLD is NP-complete (even without the above reduction).

(6 points)
2.) (a) Consider the following clause set $\delta(\varphi)$ which has been derived from an (unknown) formula $\varphi$ by an improved version of Tseitin’s translation (atoms have not been labeled).

\[
C_1: \ell_1 \lor \neg x \lor \neg y \\
C_2: \neg \ell_1 \lor x \\
C_3: \neg \ell_1 \lor y \\
C_4: \neg \ell_2 \lor \neg y \lor z \\
C_5: \ell_2 \lor y \\
C_6: \ell_2 \lor \neg z \\
C_7: \neg \ell_3 \lor \neg \ell_1 \lor z \\
C_8: \ell_3 \lor \ell_1 \\
C_9: \ell_3 \lor \neg z \\
C_{10}: \neg \ell_4 \lor \neg x \lor \ell_2 \\
C_{11}: \ell_4 \lor x \\
C_{12}: \ell_4 \lor \neg \ell_2 \\
C_{13}: \neg \ell_5 \lor \neg \ell_4 \lor \ell_3 \\
C_{14}: \ell_5 \lor \ell_4 \\
C_{15}: \ell_5 \lor \neg \ell_3
\]

(i) Reconstruct $\varphi$ from $\delta(\varphi)$.
(ii) Prove the correctness of the propositional resolution rule.
(iii) Prove the validity of $\varphi$ by resolution (no additional translation to normal form is allowed!). You are allowed to add a single unit clause (i.e., a clause containing exactly one literal). Please explain your approach!

(7 points)

(b) Clarify the logical status of each of the following formulas. If one is $T_{\text{cons}}^E$-valid or $T_{\text{cons}}^E$-unsatisfiable, then prove it using the semantic argument method. If one is $T_{\text{cons}}^E$-satisfiable but not $T_{\text{cons}}^E$-valid, then present a satisfying and a falsifying interpretation. Argue formally that the formula evaluates to true resp. false under the constructed interpretations.

\[
\varphi_0: \text{cons(car}(x), \text{cdr}(x)) = \text{cons}(y, z) \land \text{cons(car}(x), \text{cdr}(x)) \neq x \\
\rightarrow x \neq \text{cons}(y, z)
\]

\[
\varphi_1: \neg \text{atom}(x) \land \text{car}(x) = y \land \text{cdr}(x) = z \land x \neq \text{cons}(y, z)
\]

Besides the equality axioms, the following axioms of $T_{\text{cons}}^E$ may be helpful.

- $\forall x, y \text{ car}(\text{cons}(x, y)) = x$ (left projection)
- $\forall x, y \text{ cdr}(\text{cons}(x, y)) = y$ (right projection)
- $\forall x \neg \text{atom}(x) \rightarrow \text{cons(car}(x), \text{cdr}(x)) = x$ (construction)
- $\forall x, y \neg \text{atom}(\text{cons}(x, y))$ (atom)

(8 points)

(15 points)
3.) (a) Let $p$ be the following IMP program loop, containing the integer-valued program variables $x, i, b$:

\[
\textbf{while } x < b \textbf{ do}
\begin{align*}
i &:= i + 1; \\
x &:= i \ast i;
\end{align*}
\textbf{od}
\]

Which of the following program assertions are inductive loop invariants of $p$?

- $I_1 : \ x - i \ast i \neq 0$
- $I_2 : \ x \leq b$
- $I_3 : \ 	ext{true}$

Give formal details justifying your answer. That is, if an assertion is an inductive loop invariant, provide a formal proof of it based on Hoare logic. If an assertion is not an inductive loop invariant, give a counterexample.

**Note:** You need to use the definition of an assertion being inductive invariant.

(6 points)

(b) Let $p$ be the following IMP program loop, containing the integer-valued program variables $i, x, y$:

\[
\textbf{while } i < 10 \textbf{ do}
\begin{align*}
x &:= x - 1; \\
i &:= i + 1; \\
y &:= y + x + i;
\end{align*}
\textbf{od}
\]

Give a loop invariant for the \textbf{while} loop in $p$ and prove the validity of the partial correctness triple \{i = 0 \land x = 10 \land y = 0\} $p\{y = 100\}$.

**Note:** Make sure that your invariant expresses equalities among $i, x, y$ as well equalities among $i, x$.

(9 points)
4.) (a) Consider the Kripke structures $M_1$ and $M_2$. The initial state of $M_1$ is $s_0$, the initial state of $M_2$ is $t_0$.

**Kripke structure $M_1$:**

- $s_0: \{c\}$
- $s_1: \{a\}$
- $s_2: \{a\}$
- $s_3: \{b\}$
- $s_4: \{b\}$

**Kripke structure $M_2$:**

- $t_0: \{c\}$
- $t_1: \{a\}$
- $t_2: \{c\}$
- $t_3: \{b\}$
- $t_4: \{b\}$
- $t_5: \{b\}$

i. Briefly explain why $M_2$ does not simulate $M_1$.

ii. Add a minimal set of edges to $M_2$ such that the extended Kripke structure $M'_2$ simulates $M_1$. Provide the additional edges and a non-empty simulation relation $H$ that witnesses $M_1 \preceq M'_2$.

(4 points)
(b) Consider the following Kripke structure $M$:

For each of the following formulae $\varphi$,

i. indicate whether the formula is in CTL, LTL, and/or CTL*, and

ii. list the states $s_i$ on which the formula $\varphi$ holds; i.e. for which states $s_i$ do we have $M, s_i \models \varphi$?

(If $\varphi$ is a path formula, list the states $s_i$ such that $M, s_i \models A \varphi$.)

<table>
<thead>
<tr>
<th>$\varphi$</th>
<th>CTL</th>
<th>LTL</th>
<th>CTL*</th>
<th>States $s_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{EF}(c)$</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
</tr>
<tr>
<td>$c \mathbf{U} b$</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
</tr>
<tr>
<td>$\mathbf{AX}(c)$</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
</tr>
<tr>
<td>$\mathbf{G}(a)$</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
</tr>
<tr>
<td>$\mathbf{AF}(a \land c)$</td>
<td>□</td>
<td>□</td>
<td>□</td>
<td>□</td>
</tr>
</tbody>
</table>

(5 points)
(c) **LTL tautologies**

Prove that the following formulas are tautologies, i.e., they hold for every Kripke structure $M$ and every path $\pi$ in $M$, or find a Kripke structure $M$ and path $\pi$ in $M$, for which the formula does not hold and justify your answer.

i. $\neg Gp \land Fq \Rightarrow p \ U (q \lor \neg p)$

ii. $p \ U (q \lor \neg p) \Rightarrow \neg Gp \land Fq$

(6 points)