1	2	3	4	Σ	Grade

6.0/4.0 VU Formale Methoden der Informatik
185.291 June 25, 2021

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1.) Given an undirected graph $G = (V, E)$, we call a set $S \subseteq V$ *self-defending* in G if for each $v \in S$ and $u \in V$ with $(u, v) \in E$, there exists a $w \in S$ with $(w, u) \in E$.

Consider the following decision problem:

> **SELF-DEFENSE(SD)**
>
> INSTANCE: A directed graph $G = (V, E)$ and two vertices $a, b \in V$.
>
> QUESTION: Does there exist a set self-defending set $S \subseteq V$ in G with $a \in S$ and $b \notin S$.

(a) The following function f provides a polynomial-time many-one reduction from **SD** to **SAT**: for an instance $I = ((V, E), a, b)$ of **SD** let $f(I) = \varphi$ over atoms x_v $(v \in V)$:

$$\varphi \quad = \quad x_a \wedge \neg x_b \wedge \bigwedge_{v \in V} (\neg x_v \vee \bigwedge_{(u,v) \in E} \bigvee_{(w,u) \in E} x_w)$$

It holds that I is a yes-instance of **SD** $\Longleftrightarrow$ $f(I)$ is a yes-instance of **SAT**.

Show the $\Longrightarrow$ direction of the statement.

(9 points)

(b) In what follows assume the reduction from **SD** to **SAT** is correct, and recall that **SAT** is NP-complete.

Tick the correct statements (for ticking a correct statement a certain number of points is given; ticking an incorrect statement results in a substraction of the same amount; you cannot go below 0 points):

- ○ **SD** is NP-complete
- ○ **SD** is NP-hard
- ○ **SD** is in NP
- ○ **SD** is in NP, but not in P
- ○ for any NP-complete problem, there exists a polynomial-time many-one reduction from **SD** to that problem
- ○ for any NP-complete problem, there exists a polynomial-time many-one reduction from that problem to **SD**

(6 points)

2.) We consider the function P which was introduced by Rózsa Péter in 1935.

Algorithm 1: The function P

Input: x, y, two *non-negative* integers
Output: The computed integer value for x, y

if $x == 0$ **then**
  return $2y + 1$;
else if $y == 0$ **then**
  return $\mathrm{P}(x-1, 1)$;
else return $\mathrm{P}(x-1, \mathrm{P}(x, y-1))$;

(a) Let $\mathbb{N}_0$ denote the set of natural numbers *including* 0. Use well-founded induction to show

$$\forall x\, \forall y\, \big((x \in \mathbb{N}_0 \wedge y \in \mathbb{N}_0) \;\rightarrow\; \mathrm{P}(x,y) > x + y\big).$$

(12 points)

(b) Suppose $\texttt{P}_{\texttt{C}}$ is an implementation of P in the `C` programming language with x and y of type unsigned integers of size 32 bit (i.e., of type `uint32_t`). Is

$$\mathrm{P}(x', y') = \texttt{P}_{\texttt{C}}(x', y')$$

true for all integers x', y' satisfying $0 \leq x', y' \leq$ `UINT32_MAX`, where `UINT32_MAX` is the largest value for a variable of type `uint32_t`?

If so, then prove this fact. Otherwise provide a counterexample with an exact explanation of what is computed and what is happening. **(3 points)**

3.) (a) Let p be the following IMP program:

$$
\begin{array}{l}
x := 0; y := 0; \\
\textbf{while } y < n \textbf{ do} \\
\quad x := x - 6 * y - 3; \\
\quad y := y + 1 \\
\textbf{od}
\end{array}
$$

Give a loop invariant and variant for the **while** loop in p and prove the validity of the total correctness triple $[n = 10]\ p\ [x + 300 = 0]$.

(10 points)

(b) Let p be the following IMP program:

$$
\begin{array}{l}
\textbf{if } x < y \textbf{ do} \\
\quad a := y; \\
\quad y := x; \\
\quad x := a; \\
\textbf{od} \\
x := y - x; \\
z := z + x * y
\end{array}
$$

Given the program p above, is it true, that the triple $\{A\}\ p\ \{B\}$ is valid if and only if $VC(p, B) \wedge (A \implies wlp(p, B))$? Briefly justify your answer.

(2 points)

(c) Let p be the following IMP program containing an integer-valued program variable x:

$$
\begin{array}{l}
i := 0 \\
\textbf{while } x > 0 \textbf{ do} \\
\quad i := i + 1 \\
\textbf{od}
\end{array}
$$

Consider the invalid Hoare triple $\{true\}\ p\ \{false\}$. Which of the following counterexamples is correct? Tick all the boxes of program states denoting valid counterexamples to the above triple. [Each correct box counts one point, that is you can lose points for incorrectly ticking or leaving the box empty with a minimum of 0 points. You will not lose points for other exercises.]

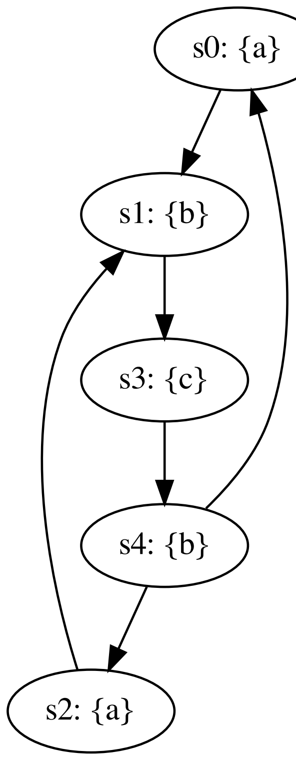
☐ $\sigma(x) = 0$ ☐ $\sigma(x) = 1$ ☐ $\sigma(x) = -1$

(3 points)

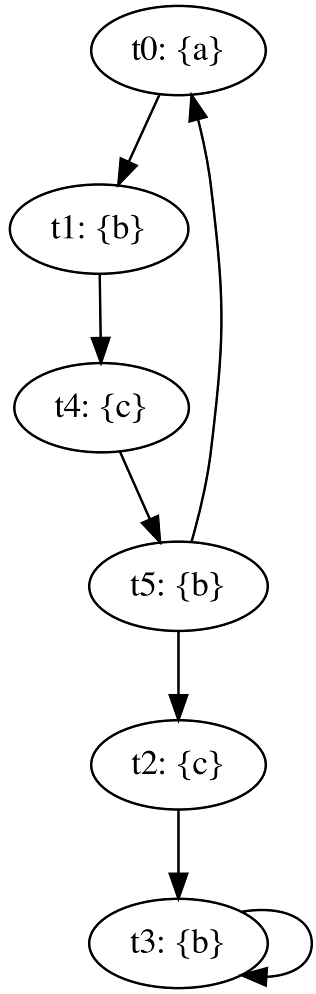
4.) (a) **Simulation**

Provide a non-empty simulation relation H that witnesses $M_1 \leq M_2$, where M_1 and M_2 are shown below. The initial state of M_1 is s_0, the initial state of M_2 is t_0:

Kripke structure M_1:



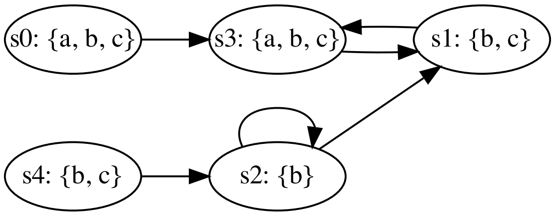
Kripke structure M_2:



(4 points)

(b) **CTL Marking Algorithm**

Consider the following Kripke structure M:



Execute the **CTL Marking Algorithm** to determine which states s_i satisfy the formulae Φ

i. $\mathbf{EXEX}a$, and
ii. $\mathbf{AF}(\neg c)$.

In particular,

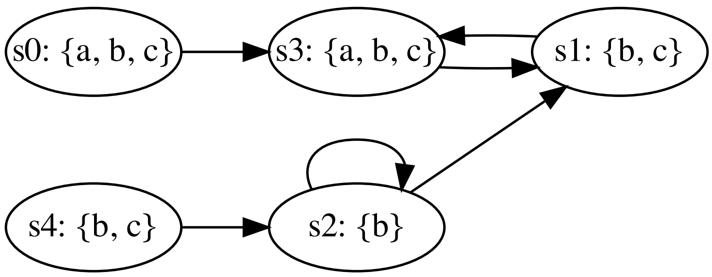
i. Transform Φ into an equivalent formula Φ' in the *existential fragment* of CTL.
ii. List the subformulae of Φ'.
iii. For increasing nesting depth i, iteratively give the states s_i marked by subformulae $\phi_0, \psi_0, \phi_1, \psi_1, \ldots$ of Φ'.
iv. Finally, give the return value of the Marking Algorithm. That is, list the states s_i that satisfy formula Φ, i.e., for which states do we have that $M, s_i \models \Phi$?

Hint: Recall that the algorithm starts by marking propositional atoms ϕ_0. It then iteratively marks boolean combinations ψ_i of subformulas ϕ_i, and temporal operator applications $\phi_{i+1} = \circ\ \psi_i$ where $\circ \in \{\mathbf{EF}, \mathbf{EU}, \mathbf{EG}, \mathbf{EX}\}$.

i) Answer template for $\Phi = \mathbf{EXEX}a$

Subformulae of Φ: ______________________________

Annotate the states of M with the subformulae by which the Marking Algorithm marks them:



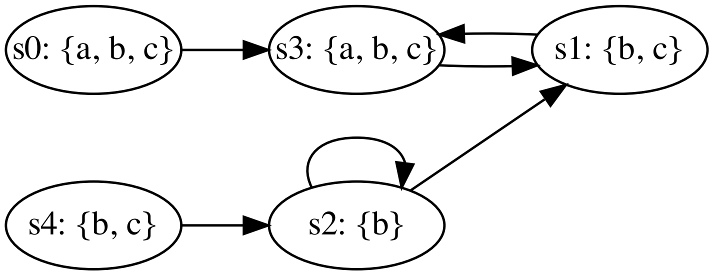
States satisfying Φ: ______________________________

ii) Answer template for $\Phi = \mathbf{AF}(\neg c)$

Equivalent existential formula $\Phi' \equiv \Phi$: ______________________________

Subformulae of Φ': ____

Annotate the states of M with the subformulae by which the Marking Algorithm marks them:



States satisfying Φ: ____

(7 points)

(c) **LTL tautologies**

Prove that the following formulas are tautologies, i.e., they hold for every Kripke structure M and every path π in M, or find a Kripke structure M and path π in M, for which the formula does not hold and justify your answer.

i. $(\mathbf{F}a) \wedge (\mathbf{F}b) \Rightarrow \mathbf{F}(a \wedge b)$

ii. $(\mathbf{G}a) \vee (\mathbf{G}b) \Rightarrow \mathbf{G}(a \vee b)$

(4 points)