1.) (a) Recall the HALTING problem which takes a program and a string as input, and consider the following variant thereof:

HALTING-X

INSTANCE: Two program $\Pi_1, \Pi_2$ that take a string as input.

QUESTION: Is it true, that for all input strings $I$: if $\Pi_1$ halts on $I$ then $\Pi_2$ halts on $I$.

The following function $f$ provides a polynomial-time many-one reduction from HALTING to HALTING-X: for a program $\Pi$ and a string $I$ let $f(\Pi, I) = (\Pi_1, \Pi_2)$ with

\[
\Pi_1(string \ S) = \text{return};  \\
\Pi_2(string \ S) = \begin{cases} 
\text{call } \Pi(S); & \text{ if } (S = I) \\
\text{return} & \text{ otherwise}
\end{cases}
\]

Show that $(\Pi, I)$ is a yes-instance of HALTING $\iff (\Pi_1, \Pi_2)$ is a yes-instance of HALTING-X.

(9 points)

(b) Tick the correct statements (for ticking a correct statement a certain number of points is given; ticking an incorrect statement results in a substraction of the same amount; you cannot go below 0 points):

- Since HALTING is undecidable, our reduction from (a) shows that HALTING-X is undecidable.
- Since HALTING is semi-decidable, our reduction from (a) shows that HALTING-X is semi-decidable.
- Since HALTING is undecidable, our reduction from (a) shows that there is no SIMPLE program that solves HALTING-X.
- Since HALTING is semi-decidable, our reduction from (a) shows that there is a SIMPLE program that solves HALTING-X.
- If we would have a decision procedure for HALTING-X, we can solve HALTING using our reduction from (a).
- If we would have a decision procedure for HALTING, we can solve HALTING-X using our reduction from (a).

(6 points)
2.) (a) Consider the theory $\mathcal{T}_A$ of arrays and the following formula

$$\varphi: a(\ell \triangleleft v)[k] \triangleq b[k] \land b[k] \neq v \land a[k] \triangleq v \land (\forall i\ (i \neq \ell \rightarrow a[i] \triangleq b[i])).$$

If $\varphi$ is $\mathcal{T}_A$-sat, then provide a $\mathcal{T}_A$-model for $\varphi$. For the proposed model, you have to show that it satisfies all axioms of $\mathcal{T}_A$ and $\varphi$.

If $\varphi$ is $\mathcal{T}_A$-unsat, then provide a proof in the semantic argument method (similarly to the proofs in the lecture and on the extra sheets). If you use a derived rule, you have to prove the correctness of the rule in the same method.

Besides the equality axioms reflexivity, symmetry and transitivity, you have the following ones for arrays.

- $\forall a, i, j\ (i \triangleq j \rightarrow a[i] \triangleq a[j])$ (array congruence)
- $\forall a, v, i, j\ (i \triangleq j \rightarrow a(i \triangleleft v)[j] \triangleq v)$ (read-over-write 1)
- $\forall a, v, i, j\ (i \neq j \rightarrow a(i \triangleleft v)[j] \triangleq a[j])$ (read-over-write 2)

(10 points)

(b) Apply the sparse method to the following $E$-formula

$$\psi^E: a \triangleq b \land (b \neq d \rightarrow (b \neq c \rightarrow c \neq d)) \land (d \neq c \land c \neq c \neq d)$$

and derive a short satisfiability-equivalent propositional formula consisting of the propositional skeleton and the transitivity constraints. Name each step in the sparse method and explain briefly, why you apply the step or why you don’t. (5 points)
3.) (a) Let $p$ be the following IMP program:

\[
\begin{align*}
    x &:= 0; y := 0; \\
    \textbf{while } x < n \textbf{ do} \\
    & \quad y := y + 3 \times x; \\
    & \quad x := x + 1 \\
    \textbf{od}
\end{align*}
\]

Give a loop invariant and variant for the while loop in $p$ and prove the validity of the total correctness triple $[n = 16] p [y = 360]$. 

(10 points)

(b) Provide a non-trivial pre-condition $A$ and a non-trivial post-condition $B$, such that the total correctness triple 

\[ [A] x := 1; \textbf{abort} [B] \]

is valid.

Trivial means equivalent to true or false, so your pre-condition $A$ and postcondition $B$ should not be equivalent to true or false. In case such a $A$ and/or $B$ does not exist, explain why there exist no such $A$ and/or $B$. 

(3 points)

(c) Consider the partial correctness triple

\[ \{x \geq 0\} y = 5 \times x \{y > x\} \]

Is this triple valid? If so, give a formal proof. Otherwise, give a counterexample.

(2 points)
4.) (a) Provide a non-empty simulation relation $H$ that witnesses $M_1 \leq M_2$, where $M_1$ and $M_2$ are shown below. The initial state of $M_1$ is $s_0$, the initial state of $M_2$ is $t_0$:

**Kripke structure $M_1$:**

- $s_0$: {c}
- $s_1$: {b}
- $s_2$: {b}
- $s_3$: {a}
- $s_4$: {a}

**Kripke structure $M_2$:**

- $t_0$: {c}
- $t_1$: {c}
- $t_2$: {b}
- $t_3$: {b}
- $t_4$: {b}
- $t_5$: {a}

(4 points)
(b) Consider the following Kripke structure $M$:

![Kripke structure diagram]

For each of the following formulae $\varphi$,

i. indicate whether the formula is in CTL, LTL, and/or CTL*, and

ii. list the states $s_i$ on which the formula $\varphi$ holds; i.e. for which states $s_i$ do we have $M, s_i \models \varphi$?

(If $\varphi$ is a path formula, list the states $s_i$ such that $M, s_i \models A\varphi$.)

<table>
<thead>
<tr>
<th>$\varphi$</th>
<th>CTL</th>
<th>LTL</th>
<th>CTL*</th>
<th>States $s_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G(a)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$AX(c)$</td>
<td></td>
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<tr>
<td>$E[(c) U (a)]$</td>
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<tr>
<td>$AG(a \land b)$</td>
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<tr>
<td>$F(c)$</td>
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</tbody>
</table>

(5 points)
(c) **LTL tautologies**

Prove that the following formulas are tautologies, i.e., they hold for every Kripke structure $M$ and every path $\pi$ in $M$, or find a Kripke structure $M$ and path $\pi$ in $M$, for which the formula does not hold and justify your answer.

i. $(G_a \Rightarrow GFb) \Rightarrow G(G_a \Rightarrow Fb)$

ii. $G(G_a \Rightarrow Fb) \Rightarrow (G_a \Rightarrow GFb)$

(6 points)