1	2	3	4	Σ	Grade

6.0/4.0 VU Formale Methoden der Informatik

185.291 **January, 29 2021**

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1.) Recall the NP-complete problem **SAT** and its specialization **3SAT** which is also NP-complete:

> **3SAT**
>
> INSTANCE: A propositional formula φ in 3-CNF, i.e. of the form $\bigwedge_{i=1}^{n}(l_{i1} \vee l_{i2} \vee l_{i3})$.
>
> QUESTION: Does there exists a truth assignment T that makes φ true?

Now consider the following further restriction:

> **3SATX**
>
> INSTANCE: A propositional formula φ in 3-CNF, where each variable occurs positively at most two times (i.e., at most two times a variable is not in the scope of negation).
>
> QUESTION: Does there exists a truth assignment T that makes φ true?

(a) The following function f provides a polynomial-time many-one reduction from **3SAT** to **3SATX**: for a formula $\varphi = \bigwedge_{i=1}^{n}(l_{i1} \vee l_{i2} \vee l_{i3})$ over variables V let

$$\begin{aligned} f(\varphi) \quad = \Big(& \bigwedge_{v \in V} \big((\neg v \vee \neg v \vee \neg \bar{v}) \wedge (v \vee v \vee \bar{v})\big) \wedge \\ & \bigwedge_{i=1}^{n} (l^*_{i1} \vee l^*_{i2} \vee l^*_{i3})\Big) \end{aligned}$$

where $l^*_{ij} = \neg v$ if $l_{ij} = \neg v$ and $l^*_{ij} = \neg \bar{v}$ if $l_{ij} = v$ (i.e., we replace each literal v in φ by $\neg \bar{v}$ for all $v \in V$).

It can be shown that φ is a yes-instance of **3SAT** $\Longleftrightarrow$ $f(\varphi)$ is a yes-instance of **3SATX**. Provide a proof for the $\Longrightarrow$ direction.

(9 points)

(b) Tick the correct statements (for ticking a correct statement a certain number of points is given; ticking an incorrect statement results in a substraction of the same amount; you cannot go below 0 points):

- Since **3SAT** is NP-hard, our reduction from (a) shows that **3SATX** is in NP.
- Since **3SAT** is NP-hard, our reduction from (a) shows that **3SATX** is NP-hard.
- Since **3SAT** is in NP, our reduction from (a) shows that **3SATX** is NP-hard.
- Since **3SATX** is a special case of **SAT**, **3SATX** must be contained in NP.
- Since **3SATX** is a special case of **3SAT**, **3SATX** must be contained in NP.
- Since **3SATX** is a special case of **3SAT**, **3SATX** must be NP-hard.

(6 points)

2.) (a) We consider the theory $\mathcal{T}_A$ of arrays from the lecture.

i. What is the signature of this theory?
ii. What kinds of axioms are available in this theory? Please name them.
iii. Consider a $\mathcal{T}_A$-formula ψ and suppose that ψ is not valid. What is a counter-example to $\mathcal{T}_A$-validity of ψ and what properties has this counter-example to satisfy?

(4 points)

(b) Consider the theory $\mathcal{T}_A$ of arrays and the following formula

$$\varphi: \quad a[i] \not\doteq v \rightarrow \big(\exists j \; a[j] \not\doteq b\langle i \triangleleft v\rangle[j]\big) \; .$$

If φ is $\mathcal{T}_A$-valid, then provide a proof in the semantic argument method (similarly to the proofs in the lecture and on the extra sheets). If φ is not $\mathcal{T}_A$-valid, then provide a counter-example.

Besides the equality axioms reflexivity, symmetry and transitivity, you have the following ones for arrays.

- $\forall a, i, j \; \big(i \doteq j \;\rightarrow\; a[i] \doteq a[j]\big)$ (array congruence)
- $\forall a, v, i, j \; \big(i \doteq j \;\rightarrow\; a\langle i \triangleleft v\rangle[j] \doteq v\big)$ (read-over-write 1)
- $\forall a, v, i, j \; \big(i \not\doteq j \;\rightarrow\; a\langle i \triangleleft v\rangle[j] \doteq a[j]\big)$ (read-over-write 2)

Please be precise. In a proof indicate exactly why proof lines follow from some other(s) and name the used rule. If you use derived rules you have to prove them. **(11 points)**

3.) (a) Let p be the following program:

$$
\begin{array}{l}
x := 0; z := 0; y := 0; \\
\textbf{while } x < n \textbf{ do} \\
\quad y := y + 3; \\
\quad z := z + 5; \\
\quad x := x + 1 \\
\textbf{od}
\end{array}
$$

Give a loop invariant for the **while** loop in p and prove the validity of the partial correctness triple $\{n > 1\}\ p\ \{z - y = 2 * n\}$.

(9 points)

(b) Let p be the following program:

$$
\begin{array}{l}
n := 0 \\
\textbf{while } x > 0 \wedge y > 0 \textbf{ do} \\
\quad \textbf{if } n = 0 \textbf{ then} \\
\quad\quad y := y + 1; \\
\quad\quad x := x - 2; \\
\quad \textbf{else} \\
\quad\quad x := x + 1 \\
\quad\quad y := y - 2 \\
\quad n := 1 - n \\
\textbf{od}
\end{array}
$$

Provide a loop variant t for the **while** loop in p strong enough to prove the validity of the total correctness triple $[x \geq 0 \wedge y \geq 0]\ p\ [x \leq 0 \vee y \leq 0]$. You may assume the invariant to be *true*.

You are **not** required to write a proof here, just state a suitable variant.

(2 points)

(c) Is the following theorem correct?

"For all assertions A, B and programs p, it holds that $[A]\ p\ [B]$ is valid if and only if $(VC(p, B)\ \wedge\ (A\ \Rightarrow\ wp\,(p, B))\ )$. "

If it is, give an argument why. If not, what is wrong?

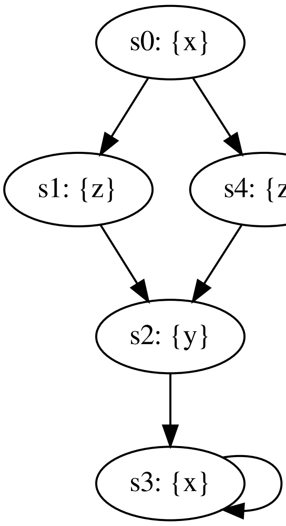
Be concise and write no more than 1-2 sentences. **(2 points)**

(d) Let n, m be integer-valued constants and A an assertion. Is there a state σ and a program p such that $\sigma \models \{true\}\ \texttt{p}\ \{false\}$? If so, provide such a state σ and program p. If, not, explain why there exists no such σ.

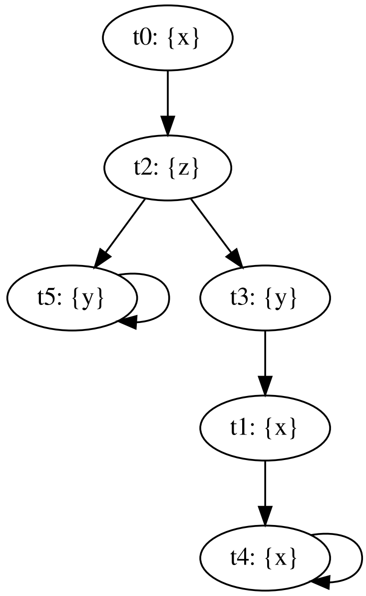
(2 points)

4.) (a) Provide a non-empty simulation relation H that witnesses $M_1 \leq M_2$, where M_1 and M_2 are shown below. The initial state of M_1 is s_0, the initial state of M_2 is t_0:

Kripke structure M_1:

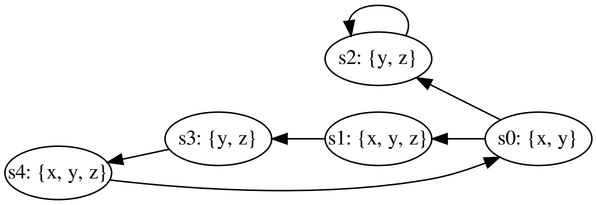


Kripke structure M_2:



(4 points)

(b) Consider the following Kripke structure M:



For each of the following formulae φ,

i. check the respective box if the formula is in CTL, LTL, and/or CTL*, and

ii. list the states s_i on which the formula φ holds; i.e. for which states s_i do we have $M, s_i \models \varphi$?
Hint: If φ is a path formula, list the states s_i such that $M, s_i \models \mathbf{A}\varphi$.

φ	CTL	LTL	CTL*	States s_i
$\mathbf{E}[(z)\ \mathbf{U}\ (x)]$	☐	☐	☐	
$\mathbf{X}(x)$	☐	☐	☐	
$\mathbf{AF}(y)$	☐	☐	☐	
$\mathbf{G}(z)$	☐	☐	☐	
$\mathbf{F}(x \wedge z)$	☐	☐	☐	

(5 points)

(c) **LTL tautologies**

Prove that the following formulas are tautologies, i.e., they hold for every Kripke structure M and every path π in M, or find a Kripke structure M and path π in M, for which the formula does not hold and justify your answer.

i. $((\mathbf{GF}x) \Rightarrow (\mathbf{GF}y)) \Rightarrow \mathbf{G}(x \Rightarrow \mathbf{F}y)$

ii. $\mathbf{G}(x \Rightarrow \mathbf{F}y) \Rightarrow ((\mathbf{GF}x) \Rightarrow (\mathbf{GF}y))$

(6 points)