1.) Provide a reduction from 2-COLORABILITY to 3-COLORABILITY, and prove that your reduction is correct.

(15 points)

Hint: For the reduction it suffices to suitably introduce one additional vertex to the input graph.

We recall the definitions of 2-COLORABILITY and 3-COLORABILITY:

2-COLORABILITY
INSTANCE: An undirected graph $G = (V, E)$.
QUESTION: Does there exist a function $\mu$ from vertices in $V$ to values in $\{1, 2\}$ such that $\mu(v_1) \neq \mu(v_2)$ for any edge $[v_1, v_2] \in E$.

3-COLORABILITY
INSTANCE: An undirected graph $G = (V, E)$.
QUESTION: Does there exist a function $\mu$ from vertices in $V$ to values in $\{1, 2, 3\}$ such that $\mu(v_1) \neq \mu(v_2)$ for any edge $[v_1, v_2] \in E$.

2.) (a) Clarify the logical status of each of the following formulas. If one is $T^{E}_{\text{cons}}$-valid or $T^{E}_{\text{cons}}$-unsatisfiable, then prove it using semantics. If one is $T^{E}_{\text{cons}}$-satisfiable but not $T^{E}_{\text{cons}}$-valid, then present a satisfying and a falsifying interpretation. Argue formally that the formula evaluates to true resp. false under the constructed interpretations.

i. $\varphi_0: \text{cons}(\text{car} (x), \text{cdr} (x)) = \text{cons}(y, z) \land \text{cons}(\text{car} (x), \text{cdr} (x)) \neq x \rightarrow x \neq \text{cons}(y, z)$

ii. $\varphi_1: \neg \text{atom} (x) \land \text{car} (x) \neq y \land \text{cdr} (x) \neq z \land x \neq \text{cons}(y, z)$

iii. $\varphi_2: \text{car} (x) \neq y \land \text{cdr} (x) \neq z \land x \neq \text{cons}(y, z)$

Besides the equality axioms, the following axioms of $T^{E}_{\text{cons}}$ may be helpful.

- $\forall x, y \text{ car}(\text{cons}(x, y)) \doteq x$ (left projection)
- $\forall x, y \text{ cdr}(\text{cons}(x, y)) \doteq y$ (right projection)
- $\forall x, y \neg \text{atom} (x) \rightarrow \text{cons}(\text{car} (x), \text{cdr} (x)) \doteq x$ (construction)
- $\forall x, y \neg \text{atom}(\text{cons}(x, y))$ (atom)

(12 points)

(b) Show that the propositional resolution rule is sound. (3 points)

3.) Verify that the following program doubles the value of $x$, i.e., that $x$ contains two times its initial value when the program terminates. For which inputs does it terminate? Choose appropriate pre- and postconditions and show that the assertion is totally correct.

Hint: Use $y = 2x_0 + x$ as a starting point for the invariant, where $x_0$ denotes the initial value of $x$. You may have to extend the formula to prove termination.

Remember the annotation rule

\[
\text{while } e \text{ do } \cdots \text{ od } \mapsto \{ \text{Inv} \} \text{ while } e \text{ do } \{ \text{Inv} \land e \land t=t_0 \} \cdots \{ \text{Inv} \land (e \Rightarrow 0 \leq t<t_0) \} \text{ od } \{ \text{Inv} \land \neg e \}
\]
\[ y := 3x; \]
\[ \text{while } 2x \neq y \text{ do} \]
\[ x := x + 1; \]
\[ y := y + 1; \]
\[ \text{od} \]

(15 points)

4. (a) Provide a non-empty simulation relation \( H \) that witnesses \( M_1 \leq M_2 \), where \( M_1 \) and \( M_2 \) are shown below. The initial state of \( M_1 \) is \( s_0 \), the initial state of \( M_2 \) is \( t_0 \):

\[ \text{Kripke structure } M_1: \]

\[ \text{Kripke structure } M_2: \]

(b) Consider the following Kripke structure \( M \):

For each of the following formulae \( \varphi \),

i. check the respective box if the formula is in CTL, LTL, and/or CTL*, and

ii. list the states \( s_i \) on which the formula \( \varphi \) holds; i.e. for which states \( s_i \) do we have \( M, s_i \models \varphi \)?
\( \varphi \) | CTL | LTL | CTL* | States \( s_i \)
---|---|---|---|---
\(((b \land c) \mathbf{U} (a))\) & & & \\
\textbf{AX}(a) & & & \\
\textbf{EG}(b) & & & \\
\textbf{EF}(a) & & & \\
\textbf{E}[((a) \mathbf{U} (a))] & & & 

(5 points)

(c) **CTL Model Checking Algorithm**

Let \( K = (S, T, L) \) be a Kripke structure and let \( p, q \) be atomic propositions. Give an algorithm that computes the set of all states \( s \in S \) that satisfy \( A[pUq] \).

(6 points)