1.) Consider the following problem:

**BOTH-HALT**

**INSTANCE:** A triple \((\Pi_1, \Pi_2, I)\), where \(I\) is a string and \(\Pi_1, \Pi_2\) are programs that take a string as input.

**QUESTION:** Is it true that \(\Pi_1\) halts on \(I\) and \(\Pi_2\) halts on \(I\)?

Provide a reduction from **BOTH-HALT** to **HALTING**. Argue formally that your reduction is correct. 

(15 points)

2.) (a) Prove or refute the following EUF-formula \(\varphi^{EUF}\):

\[
F(F(F(a))) \equiv F(a) \land F(F(a)) \equiv a \rightarrow F(a) \equiv a
\]

In case \(\varphi^{EUF}\) is valid, give a proof. Otherwise give a counterexample, i.e., an EUF-interpretation \(I\) which falsifies \(\varphi^{EUF}\). Argue formally that \(\varphi^{EUF}\) is false under \(I\).

(10 points)

(b) Apply Ackermann’s reduction to the following EUF-formula \(\psi\):

\[
p(a, F(b)) \land F(F(c)) \equiv G(G(b)) \rightarrow p(a, c)
\]

Hint: Treat uninterpreted predicates correctly. 

(4 points)

(c) Let \(\varphi\) be a formula, let \(I\) be an interpretation for \(\varphi\), and let \(M\) be a model of \(\varphi\).

Explain what \(I\) and \(M\) have in common. Explain the difference between a model and an interpretation. Is it possible that \(I\) is equal to \(M\)?

(1 point)

3.) (a) Show that the axioms \(\{ G[v/e] \} v := e \{ G \} \) and \(\{ F \} v := e \{ \exists v' (F[v/v'] \land v = e[v/v']) \}\) are equivalent, i.e., that a complete calculus needs only one of the axioms.

(7 points)

(b) Show that the following program terminates, if we assume that \(b = (c+1)^3 \land 0 \leq c^3 \leq a\) is a loop invariant.

Remember the annotation rule: \(\text{while } e \text{ do } \cdots \text{ od} \rightarrow \{ \text{Inv} \} \text{while } e \text{ do } \{ \text{Inv} \land e \land t = t_0 \} \cdots \{ \text{Inv} \land \left( e \rightarrow 0 \leq t < t_0 \right) \} \text{ od} \{ \text{Inv} \land \neg e \}\)

\[
b := 1; c := 0;
\]

\[
\text{while } b \leq a \text{ do}
\]

\[
d := 3 \times c + 6;
\]

\[
c := c + 1;
\]

\[
b := b + c \times d + 1
\]

\(\text{od}\)

(8 points)
4.) (a) Provide a non-empty simulation relation $H$ that witnesses $M_1 \preceq M_2$, where $M_1$ and $M_2$ are shown below ($M_1$ on the left, $M_2$ on the right), the initial state of $M_1$ is $s_0$, the initial state of $M_2$ is $t_0$:

Kripke structure $M_1$:

\begin{itemize}
  \item $s_0: \{c\}$
  \item $s_1: \{b\}$
  \item $s_2: \{b\}$
  \item $s_3: \{a\}$
  \item $s_4: \{a\}$
\end{itemize}

Kripke structure $M_2$:

\begin{itemize}
  \item $t_0: \{c\}$
  \item $t_1: \{b\}$
  \item $t_2: \{a\}$
  \item $t_3: \{c\}$
  \item $t_4: \{b\}$
  \item $t_5: \{b\}$
\end{itemize}

(b) State a Kripke Structure which satisfies all of the following 4 formulae:

i. $\mathbf{GF}\neg b$

ii. $\mathbf{GF}\neg a$

iii. $\mathbf{GF}(a \land b)$

iv. $\mathbf{G}(a \mathbf{U} b)$

(4 points)

(c) State two Kripke structures $A$ and $B$ such that $B$ simulates $A$ and $A$ simulates $B$ but $A$ and $B$ are not bisimilar, i.e. $A \preceq B$ and $B \preceq A$ but not $A \equiv B$. Argue why in your example $A$ and $B$ are not bisimilar.

(7 points)
DEFINITIONS

Let $M_1 = (S_1, I_1, R_1, L_1)$ and $M_2 = (S_2, I_2, R_2, L_2)$ be two Kripke structures.

Simulation

A relation $H \subseteq S_1 \times S_2$ is a simulation relation if for each $(s, s') \in H$ holds:

- $L_1(s) = L_2(s')$, and
- for each $(s, t) \in R_1$ there is a $(s', t') \in R_2$ such that $(t, t') \in H$.

$M_2$ simulates $M_1$ (denoted as $M_1 \leq M_2$), if there is a simulation relation $H \subseteq S_1 \times S_2$ such that

- for each initial state $s \in I_1$ there is an initial state $s' \in I_2$ with $(s, s') \in H$.

We say that $H$ witnesses the similarity of $M_1$ and $M_2$ in case $H$ is a simulation relation from $M_1$ to $M_2$ that satisfies the condition stated above.

Bisimulation

A relation $H' \subseteq S_1 \times S_2$ is a bisimulation relation if for each $(s, s') \in H'$ holds:

- $L_1(s) = L_2(s')$, and
- for each $(s, t) \in R_1$ there is a $(s', t') \in R_2$ such that $(t, t') \in H'$, and
- for each $(s', t') \in R_2$ there is a $(s, t) \in R_1$ such that $(t, t') \in H'$.

$M_1$ and $M_2$ are bisimilar (denoted as $M_1 \equiv M_2$) if there is a bisimulation relation $H' \subseteq S_1 \times S_2$ such that

- for each initial state $s \in I_1$ there is an initial state $s' \in I_2$ with $(s, s') \in H'$, and
- for each initial state $s' \in I_2$ there is an initial state $s \in I_1$ with $(s, s') \in H'$. 