6.0/4.0 VU Formale Methoden der Informatik (185.291)

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1.) Consider the following two problems:

> **3-COLORABILITY (3-COL)**
>
> INSTANCE: An undirected graph $G = (V, E)$.
>
> QUESTION: Does G have a *3-coloring*? That is, does there exist a function μ from vertices in V to values in $\{1, 2, 3\}$ such that $\mu(v_1) \neq \mu(v_2)$ for any edge $[v_1, v_2] \in E$?

> **UNDIRECTED GRAPH HOMOMORPHISM (HOM)**
>
> INSTANCE: A pair (G_1, G_2), where $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are undirected graphs.
>
> QUESTION: Does there exist a homomorphism from G_1 to G_2? That is, does there exist a function h from vertices in V_1 to vertices in V_2 such that for any edge $[v_1, v_2] \in E_1$ we also have $[h(v_1), h(v_2)] \in E_2$?

We provide next a reduction from **3-COL** to **HOM**. Let $G = (V, E)$ be an arbitrary undirected graph (i.e., an arbitrary instance of **3-COL**). From G we construct a pair (G_1, G_2) of undirected graphs. We let $G_1 = G$ and let $G_2 = (V_2, E_2)$ be as follows:

- $V_2 = \{v_1, v_2, v_3\}$, and
- E_2 consists of exactly the 3 (undirected) edges $[v_1, v_2]$, $[v_2, v_3]$ and $[v_1, v_3]$.

Task: Prove the "$\Rightarrow$" direction in the proof of correctness of the reduction, i.e., prove the following statement: If G is a positive instance of **3-COL**, then (G_1, G_2) is a positive instance of **HOM**.

Note: For any property that you use in your proof, make it perfectly clear why this property holds (using e.g. "by the problem reduction", "by assumption X", "by definition X"). **(15 points)**

2.) (a) First define the concept of a $\mathcal{T}$-interpretation. Then use it to define the following:

i. the $\mathcal{T}$-satisfiability of a formula;

ii. the $\mathcal{T}$-validity of a formula.

Additionally define the completeness of a theory $\mathcal{T}$ and give an example for a complete and an incomplete theory. **(5 points)**

(b) Prove that the following formula φ is $\mathcal{T}^{E}_{cons}$-valid:

$$\varphi : \qquad \neg atom(x) \wedge car(x) \doteq y \wedge cdr(x) \doteq z \rightarrow x \doteq cons(y, z)$$

Hints: Recall the axiom of construction in $\mathcal{T}^{E}_{cons}$:

$$\neg atom(x) \rightarrow cons(car(x), cdr(x)) \doteq x$$ **(5 points)**

(c) $\mathcal{T}^{E}_{cons}$ is a combined theory. How are $\mathcal{T}^{E}_{cons}$-satisfiability and $\mathcal{T}^{E}_{cons}$-validity of a formula φ related to the satisfiability and validity of φ with respect to $\mathcal{T}^{E}$ and $\mathcal{T}_{cons}$? **(5 points)**

3.) Let π be the program $\mathsf{while}\ j \neq n\ \mathsf{do}\ q := q + k;\ k := k + 2;\ j := j + 1\ \mathsf{od}$.

(a) Use the operator wp to compute a formula that specifies all states for which program π terminates. Note that this task determines the postcondition that you have to use. Remember that $\text{wp}(\texttt{while } e \texttt{ do } p \texttt{ od}, G) = \exists i\,(i \geq 0 \land F_i)$, where $F_0 = \neg e \land G$ and $F_{i+1} = e \land \text{wp}(p, F_i)$. **(5 points)**

(b) Use the annotation calculus to show that the assertion

$$\{\, n \geq 0 \,\}\; q := 0;\; k := 1;\; j := 0;\; \pi\, \{\, q = n^2 \,\}$$

is true regarding total correctness. Use $0 \leq j \leq n \land k = 2j + 1 \land q = j^2$ as invariant.
Remember the annotation rule
$\texttt{while } e \texttt{ do} \cdots \texttt{od} \mapsto \{\, Inv \,\}\texttt{while } e \texttt{ do} \{\, Inv{\land}e{\land}t{=}t_0 \,\} \cdots \{\, Inv{\land}(e \rightarrow 0{\leq}t{<}t_0) \,\}\texttt{od}\{\, Inv{\land}\neg e \,\}$
(10 points)

4.) Simulation

Let $M_1 = (S_1, I_1, R_1, L_1)$ and $M_2 = (S_2, I_2, R_2, L_2)$ be two Kripke structures.

Remember, a relation $H \subseteq S_1 \times S_2$ is a simulation relation if for each $(s, s') \in H$ it holds:

- $L_1(s) = L_2(s')$, and
- for each $(s, t) \in R_1$ there is a $(s', t') \in R_2$ such that $(t, t') \in H$.

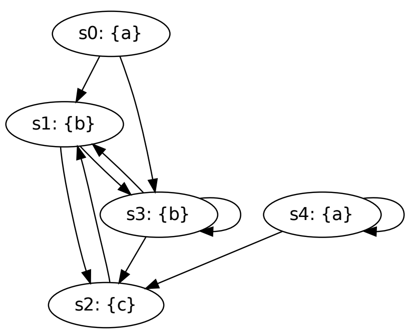
Further remember, M_2 *simulates* M_1 (denoted as $M_1 \leq M_2$), if there is a simulation relation $H \subseteq S_1 \times S_2$ *such that*

- for each initial state $s \in I_1$ there is an initial state $s' \in I_2$ with $(s, s') \in H$.

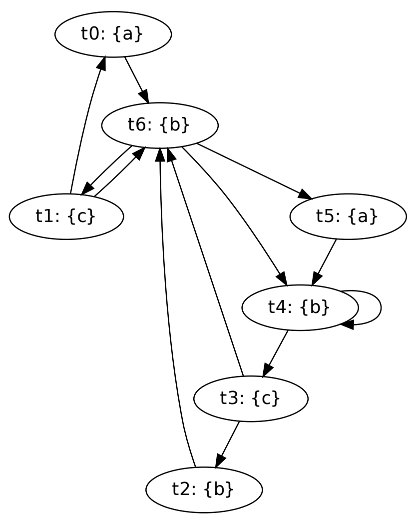
In the following, we say that H *witnesses the similarity of* M_1 *and* M_2 in case H is a simulation relation from M_1 to M_2 that satisfies the condition stated above.

(a) Provide a non-empty simulation relation H that witnesses $M_1 \leq M_2$, where M_1 and M_2 are shown below (M_1 on the left, M_2 on the right), the initial state of M_1 is s_0, the initial state of M_2 is t_0:

Kripke structure M_1:



Kripke structure M_2:



(4 points)

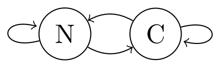
(b) Consider Kripke structure M_2 from Exercise (a).
Determine on which states t_i the following **LTL** formulae hold:

i. **F**c
ii. **G**(b ∨ c)

iii. $\mathbf{G}(\mathbf{F}\mathrm{b})$
iv. $\mathbf{G}(\mathrm{b} \rightarrow (\mathbf{X}\mathrm{a} \rightarrow \mathbf{X}\mathrm{b}))$
v. $\mathrm{a}\mathbf{U}(\mathrm{b}\mathbf{U}\mathrm{c})$

(5 points)

(c) **Background.** Consider the simple model of a process on the right: The process is either in state N or in state C.



Consider the system of N parallel processes P^N in which at most one process changes state at a time: We describe the system's state by counting the number of processes currently in N and C, respectively.

For example, in a system of three parallel processes P^3, if two processes are in state N, and one process is in state C, the corresponding configuration is $s := (n = 2, c = 1)$. Possible successors are $s'_1 := (n = 1, c = 2)$ and $s'_2 := (n = 3, c = 0)$.

Problem. We define the Kripke structure $M^N = \langle S_N, I_N, R_N, L_N \rangle$ corresponding to P^N:

- $S_N = I_N = \{(n, c) \mid n, c \in \{0, 1, \ldots, N\} \text{ and } n + c = N\}$
- $((n, c), (n', c')) \in R_n$ if and only if $n' = n + k, c' = c - k, k \in \{-1, 0, 1\}$ (at most one process moves at a time)
- $p \in L_N(s) \Leftrightarrow c > 0$ where the set of atomic propositions $AP = \{p\}$.

We consider the systems of three and two parallel processes P^3 and P^2. We define $H \subseteq S_3 \times S_2$ as

$$H = \{((n_1, c_1), (n_2, c_2)) \mid \min(n_1, 1) = \min(n_2, 1) \wedge \min(c_1, 1) = \min(c_2, 1)\}$$

(H encodes the idea of observing if at last one process is in the respective state.)

Show that H witnesses $M^3 \leq M^2$.

(6 points)