1.) Consider the following two problems:

**3-COLORABILITY (3-COL)**

INSTANCE: An undirected graph $G = (V, E)$.

QUESTION: Does $G$ have a 3-coloring? That is, does there exist a function $\mu$ from vertices in $V$ to values in $\{1, 2, 3\}$ such that $\mu(v_1) \neq \mu(v_2)$ for any edge $[v_1, v_2] \in E$?

**UNDIRECTED GRAPH HOMOMORPHISM (HOM)**

INSTANCE: A pair $(G_1, G_2)$, where $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are undirected graphs.

QUESTION: Does there exist a homomorphism from $G_1$ to $G_2$? That is, does there exist a function $h$ from vertices in $V_1$ to vertices in $V_2$ such that for any edge $[v_1, v_2] \in E_1$ we also have $[h(v_1), h(v_2)] \in E_2$?

We provide next a reduction from 3-COL to HOM. Let $G = (V, E)$ be an arbitrary undirected graph (i.e., an arbitrary instance of 3-COL). From $G$ we construct a pair $(G_1, G_2)$ of undirected graphs. We let $G_1 = G$ and let $G_2 = (V_2, E_2)$ be as follows:

- $V_2 = \{v_1, v_2, v_3\}$, and
- $E_2$ consists of exactly the 3 (undirected) edges $[v_1, v_2]$, $[v_2, v_3]$ and $[v_1, v_3]$.

**Task**: Prove the “⇒” direction in the proof of correctness of the reduction, i.e., prove the following statement: If $G$ is a positive instance of 3-COL, then $(G_1, G_2)$ is a positive instance of HOM.

**Note**: For any property that you use in your proof, make it perfectly clear why this property holds (using e.g. “by the problem reduction”, “by assumption $X$”, “by definition $X$”).

(15 points)

2.) (a) First define the concept of a $\mathcal{T}$-interpretation. Then use it to define the following:

i. the $\mathcal{T}$-satisfiability of a formula;
ii. the $\mathcal{T}$-validity of a formula.

Additionally define the completeness of a theory $\mathcal{T}$ and give an example for a complete and an incomplete theory. (5 points)

(b) Prove that the following formula $\varphi$ is $\mathcal{T}_{\text{cons}}^E$-valid:

$$\varphi : \neg \text{atom}(x) \land \text{car}(x) \equiv y \land \text{cdr}(x) \equiv z \rightarrow x \equiv \text{cons}(y, z)$$

Hints: Recall the axiom of construction in $\mathcal{T}_{\text{cons}}^E$:

$$\neg \text{atom}(x) \rightarrow \text{cons}(\text{car}(x), \text{cdr}(x)) \equiv x$$

(5 points)

(c) $\mathcal{T}_{\text{cons}}^E$ is a combined theory. How are $\mathcal{T}_{\text{cons}}^E$-satisfiability and $\mathcal{T}_{\text{cons}}^E$-validity of a formula $\varphi$ related to the satisfiability and validity of $\varphi$ with respect to $\mathcal{T}_{\text{cons}}^E$ and $\mathcal{T}_{\text{cons}}$? (5 points)

3.) Let $\pi$ be the program

$$\text{while } j \neq n \text{ do } q := q + k; \ k := k + 2; \ j := j + 1 \text{ od}.$$
(a) Use the operator wp to compute a formula that specifies all states for which program $\pi$ terminates. Note that this task determines the postcondition that you have to use.

Remember that $\text{wp}(\text{while } e \text{ do } p \text{ od}, G) = \exists i \geq 0 (F_i)$, where $F_0 = \neg e \land G$ and $F_{i+1} = e \land \text{wp}(p, F_i)$. (5 points)

(b) Use the annotation calculus to show that the assertion

\[ \{ n \geq 0 \} q := 0; \ k := 1; \ j := 0; \ \pi \{ q = n^2 \} \]

is true regarding total correctness. Use $0 \leq j \leq n \land k = 2j + 1 \land q = j^2$ as invariant.

Remember the annotation rule

\[
\text{while } e \text{ do } \cdots \text{ od } \mapsto \{ \text{Inv} \} \text{while } e \text{ do } \{ \text{Inv} \land e \land t = t_0 \} \cdots \{ \text{Inv} \land (e \rightarrow 0 \leq t < t_0) \} \text{ od } \{ \text{Inv} \land \neg e \}
\]

(10 points)

4.) Simulation

Let $M_1 = (S_1, I_1, R_1, L_1)$ and $M_2 = (S_2, I_2, R_2, L_2)$ be two Kripke structures.

Remember, a relation $H \subseteq S_1 \times S_2$ is a simulation relation if for each $(s, s') \in H$ it holds:

- $L_1(s) = L_2(s')$, and
- for each $(s, t) \in R_1$ there is a $(s', t') \in R_2$ such that $(t, t') \in H$.

Further remember, $M_2$ simulates $M_1$ (denoted as $M_1 \leq M_2$), if there is a simulation relation $H \subseteq S_1 \times S_2$ such that

- for each initial state $s \in I_1$ there is an initial state $s' \in I_2$ with $(s, s') \in H$.

In the following, we say that $H$ witnesses the similarity of $M_1$ and $M_2$ in case $H$ is a simulation relation from $M_1$ to $M_2$ that satisfies the condition stated above.

(a) Provide a non-empty simulation relation $H$ that witnesses $M_1 \leq M_2$, where $M_1$ and $M_2$ are shown below ($M_1$ on the left, $M_2$ on the right), the initial state of $M_1$ is $s_0$, the initial state of $M_2$ is $t_0$:

Kripke structure $M_1$:

Kripke structure $M_2$:

(b) Consider Kripke structure $M_2$ from Exercise (a).

Determine on which states $t_i$ the following LTL formulae hold:

i. $F c$

ii. $G (b \lor c)$

(4 points)
iii. \( \mathbf{G}(\mathbf{F}b) \)
iv. \( \mathbf{G}(b \rightarrow (\mathbf{X}a \rightarrow \mathbf{X}b)) \)
v. \( a\mathbf{U}(b\mathbf{U}c) \)

(5 points)

(c) **Background.** Consider the simple model of a process on the right: The process is either in state N or in state C.

Consider the system of \( N \) parallel processes \( P^N \) in which at most one process changes state at a time: We describe the system’s state by counting the number of processes currently in N and C, respectively.

For example, in a system of three parallel processes \( P^3 \), if two processes are in state N, and one process is in state C, the corresponding configuration is \( s := (n = 2, c = 1) \).

Possible successors are \( s'_1 := (n = 1, c = 2) \) and \( s'_2 := (n = 3, c = 0) \).

**Problem.** We define the Kripke structure \( M^N = \langle S^N, I^N, R^N, L^N \rangle \) corresponding to \( P^N \):

- \( S^N = I^N = \{ (n,c) \mid n,c \in \{0,1,\ldots,N \} \text{ and } n+c = N \} \)
- \( ((n,c),(n',c')) \in R^N \) if and only if \( n' = n + k, c' = c - k, k \in \{-1,0,1\} \) (at most one process moves at a time)
- \( p \in L^N(s) \iff c > 0 \) where the set of atomic propositions \( AP = \{ p \} \).

We consider the systems of three and two parallel processes \( P^3 \) and \( P^2 \). We define \( H \subseteq S^3 \times S^2 \) as

\[
H = \{ ((n_1,c_1),(n_2,c_2)) \mid \min(n_1,1) = \min(n_2,1) \land \min(c_1,1) = \min(c_2,1) \}
\]

(\( H \) encodes the idea of observing if at last one process is in the respective state.) Show that \( H \) witnesses \( M^3 \leq M^2 \).

(6 points)