6.0/4.0 VU Formale Methoden der Informatik

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1.) Consider the following problem:

ECO-VERTEX-COVER (EVC)

INSTANCE: Undirected graph $G = (V, E)$.

QUESTION: Does there exist a set N, where $N \subseteq V$, such that: for all edges $[a, b] \in E$, we have $|\{a, b\} \cap N| = 1$?

We provide next a reduction from **EVC** to **2-SAT**. Let $G = (V, E)$ be an arbitrary undirected graph (i.e. an arbitrary instance of **EVC**), where $V = \{v_1, \dots, v_n\}$. For the reduction we use propositional variables $x_1, \dots, x_n$. Then the instance φ_G of **2-SAT** resulting from G is defined as follows:

$$\varphi_G = \bigwedge_{[v_i, v_j] \in E} (x_i \vee x_j) \wedge (\neg x_i \vee \neg x_j).$$

Task: Prove the "$\Rightarrow$" direction in the proof of correctness of the reduction, i.e. prove the following statement: if G is a positive instance of **EVC**, then φ_G is a positive instance of **2-SAT**.

Note: For any property that you use in your proof, make it perfectly clear why this property holds (e.g., "by the problem reduction", "by the assumption X", "by the definition X", etc.)

(15 points)

2.) (a) We discussed in class the big picture of the SAT block. Describe in detail how a SAT solver can be employed to decide whether a given equality formula containing uninterpreted functions is valid. Explain the logical relation between the different problems in your description. **(4 points)**

(b) In the lecture, we discussed reasoning under different theories. Here we are concerned with LISP-like lists and the theory $\mathcal{T}_{cons}^{E} = \mathcal{T}_{cons} \cup \mathcal{T}_{E}$. In a verification attempt of some program, we have to prove the following:

For non-atomic lists ℓ_1, ℓ_2, if the "car" of both lists are equal and the "cdr" of both lists are equal, then ℓ_1 is equal to ℓ_2.

We formalize the above statement as follows:

$$\varphi: \quad \left[\neg atom(\ell_1) \wedge \neg atom(\ell_2) \wedge car(\ell_1) \doteq car(\ell_2) \wedge cdr(\ell_1) \doteq cdr(\ell_2)\right] \;\rightarrow\; \ell_1 \doteq \ell_2$$

Prove the statement $\mathcal{T}_{cons}^{E}$-valid, i.e., show that $\mathcal{T}_{cons}^{E} \models \varphi$.

Hint: Besides the equality axioms reflexivity, symmetry and transitivity, the following axioms from $\mathcal{T}_{cons}^{E}$ are sufficient for a proof:

(1) Substitution axioms (functional congruence) for *cons*:

$$\forall x_1 \forall x_2 \forall y_1 \forall y_2\, [(x_1 \doteq x_2 \wedge y_1 \doteq y_2) \rightarrow cons(x_1, y_1) \doteq cons(x_2, y_2)]$$

(2) Construction:

$$\forall x\, [\neg atom(x) \rightarrow cons(car(x), cdr(x)) \doteq x]$$

(11 points)

3.) (a) Consider a statement consisting only of the keyword "loopforever". When executed within a program, the program enters an infinite loop. Define the structural operational and the natural semantics of loopforever-statements. Specify the weakest precondition wp(loopforever, F), the weakest liberal precondition wlp(loopforever, F), and the strongest postcondition sp(F, loopforever) with respect to an arbitrary formula F.

(5 points)

(b) Compute the weakest precondition of the following program with respect to the postcondition $x = y$.

$$
\begin{array}{l}
y \leftarrow 0; \\
z \leftarrow x; \\
\textsf{while } z \neq 0 \textsf{ do} \\
\quad y \leftarrow y + 1; \\
\quad z \leftarrow z - 2; \\
\textsf{od}
\end{array}
$$

Remember the weakest precondition of loops: wp(while e do p od, G) $= \exists i\,(i \geq 0 \wedge F_i)$, where $F_0 = \neg e \wedge G$ and $F_{i+1} = e \wedge \text{wp}(p, F_i)$.

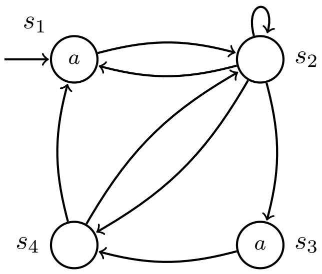
(10 points)

4.) Bisimulation.

(a) Consider two **LTL** formulas $\varphi = \mathbf{G}\,(p \rightarrow \mathbf{X}\,(\neg p \wedge q))$ and $\psi = \mathbf{G}\,\mathbf{F}\,(p \wedge \mathbf{X}\,\mathbf{X}\,(\neg p \wedge \neg q))$. Give two Kripke structures K_1 and K_2 satisfying the following:

- $K_1 \models \varphi$ and $K_1 \models \psi$;
- $K_2 \models \varphi$ and $K_2 \not\models \psi$.

(3 points)



M_1

(b) For the Kripke structure $M_1 = (S_1, I_1, R_1, L_1)$ given above, find a Kripke structure $M_2 = (S_2, I_2, R_2, L_2)$ with the following properties:

i. M_2 is bisimilar to M_1.

ii. M_2 is minimal in the number of states, that is, there is no other Kripke structure $M = (S, I, R, L)$ that is bisimilar to M_1 ($M \approx M_1$) and $|S| < |S_2|$.

Give a bisimulation relation H between M_1 and M_2.

Hint: Recall the definition of bisimulation from the lectures: M_1 and M_2 are bisimilar (in signs $M_1 \approx M_2$) iff there is a *bisimulation relation* $H \subseteq S_1 \times S_2$ with the following properties satisfied for every pair $(s_1, s_2) \in H$:

i. Labels coincide: $L_1(s_1) = L_2(s_2)$.

ii. For every transition $(s_1, t_1) \in R_1$ there is a matching transition $(s_2, t_2) \in R_2$ with $(t_1, t_2) \in H$. In the other direction, for every transition $(s_2, t_2) \in R_2$ there is a matching transition $(s_1, t_1) \in R_1$ such that $(t_1, t_2) \in H$.

iii. For every initial state $s_1 \in I_1$ there is a corresponding initial state $s_2 \in I_2$ such that $(s_1, s_2) \in H$. In the other direction, for every initial state $s_2 \in I_2$ there is a corresponding initial state $s_1 \in I_1$ with $(s_1, s_2) \in H$.

(6 points)

(c) Show that the following theorem holds.

> **Theorem.**
> Consider two Kripke structures $M_1 = (S_1, I_1, R_1, L_1)$ and $M_2 = (S_2, I_2, R_2, L_2)$ that are bisimilar, i.e., $M_1 \approx M_2$.
> Prove that for every path $s_0 s_1 \ldots s_k$ of M_1 starting with $s_0 \in I_1$ there exists a corresponding path $t_0 t_1 \ldots t_k$ of M_2 with the following properties:
>
> i. It holds that $t_0 \in I_2$.
> ii. For every $i \geq 0$ it holds that $L_1(s_i) = L_2(t_i)$.

Hint:

Recall the definition of bisimulation (see Exercise b) and use induction on the length of a path.

(6 points)