Proposition 1 (Slide 1/29) *Let $A \subseteq \mathcal{A}$ be a set of atoms, $X, Y \subseteq A$, and I an interpretation, such that $(I \cap A) = X$ and $(I \cap A') = Y'$. Then,*

1. *I is a model of $A \leq A'$ iff $X \subseteq Y$;*
2. *I is a model of $A < A'$ iff $X \subset Y$.*

ad 1) I is a model of $\bigwedge_{p \in A}(p \supset p')$ iff for each $p \in A$, $p \supset p'$ is true under I iff for each $p \in A$, $p \in I$ implies $p' \in I$ iff $(I \cap A)' \subseteq (I \cap A')$ iff $X' \subseteq Y'$ iff $X \subseteq Y$.

ad 2) I is model of $(A \leq A') \wedge \neg(A' \leq A)$ iff ($X \subseteq Y$ and not $Y \subseteq X$) iff $X \subset Y$.

Proposition 2 (Slide 2/7) *Let I, J be models of a Horn program P, then $(I \cap J)$ is a model of P.*

Towards a contradiction, suppose (i) $I \models P$ (ii) $J \models P$, and (iii) $(I \cap J) \not\models P$. From (iii), we get that there exists a rule $h \leftarrow b_1, \ldots, b_n$ in P such that each element b_i is contained in $I \cap J$, but $h \notin (I \cap J)$. Since each b_i is contained in $I \cap J$, we get that each b_i is also contained in I and in J. For $h \notin (I \cap J)$, We have two cases: (a) $h \notin I$; (b) $h \notin J$. In case (a), we immdediatly get that $I \not\models r$ (since each b_i is in I but $h \notin I$) and thus $I \not\models P$. Contradiction to (i). Likewise, in case (b), we get that $J \not\models r$ and thus $J \not\models P$. Contradiction to (ii).

Proposition 3 (Slide 2/23) *Checking whether a normal program has at least one stable model is* NP*-hard.*

Let $\phi = \bigwedge_{i=1}^{n} \bigvee_{j=1}^{m(i)} l_{ij}$ be a formula in CNF over propositional atoms V. Let, for each $v \in V$ be v' a globally new atom (representing $\neg v$). We construct:

$$\begin{aligned} \mathcal{T}[\phi] \quad = \quad & \{v \leftarrow \textit{not } v';\ v' \leftarrow \textit{not } v; \\ & \ \bot \leftarrow v, v' \mid v \in V\} \cup \\ & \{\bot \leftarrow l^{\dagger}_{i,1}, \ldots, l^{\dagger}_{i,m(i)} \mid 1 \leq i \leq n\}; \end{aligned}$$

where

- $l^{\dagger}_{i,j} = v'$ if $l_{i,j} = v$ is an atom;
- $l^{\dagger}_{i,j} = v$ if $l_{i,j} = \neg v$ is a negated atom.

Recall: SAT for CNFs is NP-hard, and observe that $\mathcal{T}[\phi]$ is constructible in polynomial from ϕ, for each CNF ϕ.

It remains to show that ϕ is satisfiable iff $\mathcal{T}[\phi]$ has at least one stable model.

Only-if: Suppose there exists an interpretation I, such that $I \models \phi$. We show that $J = I \cup (V' \setminus I')$ is stable model of $\mathcal{T}[\phi]$.

Observe that the reduct, $(\mathcal{T}[\phi])^J$, is given by

$$I \cup (V' \setminus I') \cup \tag{1}$$

$$\{\bot \leftarrow v, v' \mid v \in V\} \cup \tag{2}$$

$$\{\bot \leftarrow l^{\dagger}_{i,1}, \ldots, l^{\dagger}_{i,m(i)} \mid 1 \leq i \leq n\}. \tag{3}$$

In fact, (1) is derived by the fact that (i) rules $v \leftarrow \mathit{not}\ v'$ survive where $v' \notin J$, i.e., where $v \in I$, and the negative body is deleted; (ii) rules $v' \leftarrow \mathit{not}\ v$ survive where $v \notin I$, i.e., where $v \in J$.

First, we check that J is a classical model of $(\mathcal{T}[\phi])^J$; this holds obviously for rules (1,2). For the check-part (3), suppose $J \not\models \mathcal{T}[\phi]$. Then, for some r_i from (3), $J \not\models r_i$. Note that r_i represents the negation of the i-th clause in ϕ. Hence, if $J \not\models r_i$, I cannot satisfy the i-th clause in the CNF. This would lead to a contradiction. Hence $J \models (\mathcal{T}[\phi])^J$.

Second, we check whether J is a minimal model of $(\mathcal{T}[\phi])^J$. Clearly, no proper subset of J is a model of rules (1) of $(\mathcal{T}[\phi])^J$.

The if-direction is as follows: Suppose J is a stable model of $(\mathcal{T}[\phi])$. By the generating part (1,2), we have that, for each $v \in V$, either $v \in J$ or $v' \in J$, but not both. Since J is model of the check-part (3), no rule body is true under J; but then, each clause in ϕ is true under $J \cap V$. Hence, ϕ is satisfiable.

Proposition 4 (Slide 2/30) *Deciding whether a disjunctive program has at least a stable model is Σ_2^P-hard.*

Deciding whether a $(2,\exists)$-QBF $\Phi = \exists X \forall Y \phi$ (with ϕ a 3DNF) is true, is Σ_2^P-complete. Consider the following reduction from such $(2,\exists)$-QBFs with $\phi = \bigvee_{i=1}^{n}(l_{i,1} \wedge l_{i,2} \wedge l_{i,3})$ – with negative atoms written as $\overline{a}$ – to programs:

$$\begin{aligned}
\mathcal{T}[\Phi] \quad = \quad & \{x \vee \overline{x} \leftarrow; \bot \leftarrow x, \overline{x} \mid x \in X\} \cup && (4)\\
& \{y \vee \overline{y} \leftarrow; y \leftarrow w; \overline{y} \leftarrow w; w \leftarrow y, \overline{y} \mid y \in Y\} \cup && (5)\\
& \{w \leftarrow l_{i,1}, l_{i,2}, l_{i,3} \mid 1 \leq i \leq n\} \cup && (6)\\
& \{\bot \leftarrow \mathit{not}\ \ w\}; && (7)
\end{aligned}$$

First consider $X = \emptyset$. We show that $\Phi = \forall \phi$ is true iff $\mathcal{T}[\Phi]$ has at least a stable model. Note that $\mathcal{T}[\Phi]$ consists now only of rules (5,6,7).

To this end, let us first compute the classical models of the positive subprogram (5,6).

1. $Z = (Y \cup \overline{Y} \cup \{w\})$ is a classical model of (5,6);

2. an I with $w \notin I$, is a classical model of (5,6) iff (i) either $y \in I$ or $\overline{y} \in I$, for each $y \in Y$, and (ii) $(I \cap V)$ is not a model of ϕ.

Considering now rule (7), which forces w to be in any model, the only candidate for being answer set of $\mathcal{T}[\Phi]$ is thus Z. However, $(\mathcal{T}[\Phi])^Z$ is given by (5,6) again. Hence, Z is stable iff no I from 2. exists, i.e., iff ϕ is true under all interpretations; i.e., iff Φ is true.

Now, consider $X \neq \emptyset$. The argumentation is similar; however we now have candidates for stable models of the form $Z \cup J$ with $J \subseteq X \cup \overline{X}$, such that either $x \in J$ or $\overline{x} \in J$, for any $x \in X$. In other words, $J \cup Z$ is answer set of $\mathcal{T}[\Phi]$ iff J is a model of $\forall Y \phi$. Clearly, Φ is true iff such a J exists.

Proposition 5 (Slide 2/37) *Let P be a program over atoms V; let $J, K \subseteq V$; and let I be any interpretation, such that $(I \cap V) = J$ and $(I \cap V') = K'$. Then,*

$$I \text{ is a model of } P^* \text{ iff } K \models P^J.$$

We first show the following with I, K, J as above. Let $r \in P$. Then $I \not\models \{r\}^*$ iff $K \not\models \{r\}^J$. We have

$$\begin{array}{ll} I \not\models \{r\}^* & \text{iff} \\ (B^-(r) \cap I) = \emptyset;\ B^+(r') \subseteq I; \text{ and } (H(r') \cap I) = \emptyset & \text{iff} \\ (B^-(r) \cap J) = \emptyset;\ B^+(r) \subseteq K; \text{ and } (H(r) \cap K) = \emptyset & \text{iff} \\ K \not\models \{r\}^J. & \end{array}$$

Hence, to show the theorem, we have that $I \models P^*$ iff, for each $r \in P$, $I \models \{r\}^*$ which holds by above relation iff $K \models \{r\}^J$, for each $r \in P$, i.e., iff $K \models P^J$.

Proposition 6 (Slide 3/11) *The following propositions are equivalent:*

1. *$P \equiv_s Q$; i.e., for each program R, $\mathcal{AS}(P \cup R) = \mathcal{AS}(Q \cup R)$;*
2. *for each unary program R, $\mathcal{AS}(P \cup R) = \mathcal{AS}(Q \cup R)$;*
3. *$SE(P) = SE(Q)$.*

We show: 1. implies 2., 2. implies 3., and 3. implies 1. Clearly, 1. implies 2. holds by definition.

2. implies 3.: Indirect. Suppose $SE(P) \neq SE(Q)$. We show that 2. does not hold. Without loss of generalization, suppose there exists some $(J, I) \in SE(P)$, such that $(J, I) \notin SE(Q)$. We have two cases:

(a) $J = I$. Since $(I, I) \in SE(P)$, $I \models P$, and we have already seen that then $I \in \mathcal{AS}(P \cup I)$. On the other, $(I, I) \notin SE(Q)$ iff $I \not\models Q$. As well, we already know that $I \notin \mathcal{AS}(Q \cup I)$. Since I itself can be seen as a set of facts, I is a unary program, yielding a unary counterexample.

(b) $J \subset I$. First observe, that $(J, I) \in SE(P)$ implies $(I, I) \in SE(P)$. We have $(J, I) \notin SE(Q)$. If $(I, I) \notin SE(Q)$ we apply case (a). So suppose $(I, I) \in SE(Q)$. We already have seen that in this case, setting $R = J \cup \{p \leftarrow q \mid p, q \in (I \setminus J)\}$ yields $I \in \mathcal{AS}(Q \cup R)$. We now show that $I \notin \mathcal{AS}(P \cup R)$. To this end, observe that $J \models (P \cup R)^I = P^I \cup R$. This follows from the fact that $J \models P^I$ by assumption that $(J, I) \in SE(P)$; and by observations that $J \models J$, and $J \models \{p \leftarrow q \mid p, q \in (I \setminus J)\}$, i.e., $J \models R$. Hence, $I \notin \mathcal{AS}(P \cup R)$, with R unary.

3. implies 1.: Indirect. Without loss of generalization suppose, there exists a program R, such that $I \in \mathcal{AS}(P \cup R)$ and $I \notin \mathcal{AS}(Q \cup R)$; the other case is analogous. We show that $SE(P) \neq SE(Q)$. For $I \notin \mathcal{AS}(Q \cup R)$ we can identify two reasons:

(a) $I \not\models Q \cup R$. Then $I \not\models Q$, since $I \models R$ by assumption $I \models P \cup R$. From $I \not\models Q$, we get $(I, I) \notin SE(Q)$. On the other hand $I \models P$, and thus $(I, I) \in SE(P)$.

(b) there exists a $J \subset I$, such that $J \models (Q \cup R)^I$. Then, $J \models Q^I$ and $J \models R^I$ and from the former we get $(J, I) \in SE(Q)$. On the other hand, $J \not\models (P \cup R)^I$; otherwise I cannot be answer-set of $P \cup R$. We already know $J \models R^I$ from above. Then, $J \not\models P^I$, i.e., $(J, I) \notin SE(P)$ has to hold.

Proposition 7 (Slide 3/25) *Given disjunctive programs P, Q, and a set of atoms B, deciding $P \equiv_B Q$ is Π_3^P-hard.*

We reduce any $(\forall, 3)$-QBF $\Phi = \forall X_1 \exists X_2 \forall Y \phi$ with ϕ a 3DNF to a problem $P \equiv_B Q$, such that $P \equiv_B Q$ holds iff Φ is true. Recall, deciding the truth of a $(\forall, 3)$-QBF of this form is Π_3^P-complete. We set up the two programs as follows:

$$
\begin{aligned}
\mathcal{T}_P[\Phi] \quad &= \quad \{x \vee \overline{x} \leftarrow;\ \bot \leftarrow x, \overline{x} \mid x \in X_1 \cup X_2\} \\
&\quad\quad \{y \vee \overline{y} \leftarrow; y \leftarrow w; \overline{y} \leftarrow w; w \leftarrow y, \overline{y} \mid y \in Y\} \cup \\
&\quad\quad \{w \leftarrow l_{i,1}, l_{i,2}, l_{i,3} \mid 1 \leq i \leq n\} \cup \\
&\quad\quad \{\bot \leftarrow \mathit{not}\ \ w\}; \\
\mathcal{T}_Q[\Phi] \quad &= \quad \{x \vee \overline{x} \leftarrow;\ \bot \leftarrow x, \overline{x} \mid x \in X_1\};
\end{aligned}
$$

and $B = X_1$. Note that the problem $\mathcal{T}_P[\Phi] \equiv_B \mathcal{T}_Q[\Phi]$ is thus constructed in polynomial time from Φ.

First, the answer sets (over B) of $\mathcal{T}_Q[\Phi]$ are all sets $J \subseteq X_1$.

Second, the answer sets of $\mathcal{T}_P[\Phi]$ are easily obtained by using the argumentation for $\mathcal{T}[\Phi]$ in the proof of Proposition 4. Indeed, the two programs are the same for $X = X_1 \cup X_2$. We have seen that the answer-sets of $\mathcal{T}[\Phi]$ characterize the models of $\forall Y \phi$. Obviously, now each $J \subseteq X_1$ is included in the answer-sets of $\mathcal{T}_P[\Phi]$ iff $\mathcal{T}_P[\Phi] \equiv_B \mathcal{T}_Q[\Phi]$. On the other hand, iff each $J \subseteq X_1$ is included in the answer-sets of $\mathcal{T}_P[\Phi]$, we get that for each assignment to X_1 there exists an assignment to X_2 such that $\forall Y$ is true; i.e., $\forall X_1 \exists X_2 \forall Y \phi$ is true.

Proposition 8 (Slide 4/22) *The following propositions are equivalent:*

1. $P \equiv_s Q$;
2. *for each* $C \subseteq \mathcal{C}$, $SE(Gr(P, C)) = SE(Gr(Q, C))$;
3. *for* $D = U^+_{P \cup Q}$, $SE(Gr(P, D)) = SE(Gr(Q, D))$.

For the proof, we need further lemmas:

Lemma 1 *Let P be a program, $C, C' \subseteq \mathcal{C}$ sets of constants such that $C \subseteq C'$, and $I \subseteq B_{P,C}$.*

Then, $I \models Gr(P, C)$ iff $I \models Gr(P, C')$.

The if direction holds by the fact that $Gr(P,C) \subseteq Gr(P,C')$. The only-if direction holds in view of safety: Towards a contradiction, suppose $I \models Gr(P,C)$ but $I \not\models Gr(P,C')$. Then, there is a rule $r \in Gr(P,C') \setminus Gr(P,C)$, such that $I \not\models r$. Since $r \in Gr(P,C') \setminus Gr(P,C)$, some $c \in C' \setminus C$ occurs in r which is obtained by a variable substitution. But c has to occur in $B^+(r)$, otherwise the rule in P from which r is obtained from would not be safe. But then, since $I \subseteq B_{P,C}$, $I \not\models B^+(r)$. Therefore, $I \models r$, a contradiction.

Lemma 2 *Let P, C, C', and I as in Lemma 1.*
Then, $I \in \mathcal{AS}(Gr(P,C))$ iff $I \in \mathcal{AS}(Gr(P,C'))$.

Only-if: Since $I \in \mathcal{AS}(Gr(P,C))$, we have $I \models Gr(P,C)$ and by Lemma 1, $I \models Gr(P,C')$. Towards a contradiction, suppose some $J \subset I$ is a model of $Gr(P,C')$. Then, again by Lemma 1, $J \models Gr(P,C)$. But then, I is not answer set of $Gr(P,C)$. Contradiction. The if-direction is essentially by the same arguments.

Lemma 3 *Let $(J,I) \in SE(Gr(P,C))$ and $C' \subseteq C$.*
Then, $(J',I') \in SE(Gr(P,C'))$ with $J' = (J \cap B_{\mathcal{A},C'})$, $I' = (I \cap B_{\mathcal{A},C'})$.

Towards a contradiction, assume that $(J,I) \in SE(Gr(P,C))$, and $(J',I') \notin SE(Gr(P,C'))$. Hence, there is some $r \in Gr(P,C')$, such that $J' \not\models r^{I'}$ (this holds also for $J' = I'$, since in general $K \models r$ iff $K \models r^K$). Since $Gr(P,C') \subseteq Gr(P,C)$, $r \in Gr(P,C)$, and since r does not contain any atom from $I \setminus I'$, $r^I = r^{I'}$. Moreover, since r does not contain any atom from $J \setminus J'$, $J \not\models r^I$. Hence, $(J,I) \notin SE(Gr(P,C))$, a contradiction. This shows the lemma.

We now proceed with the proof of the proposition:

> (2) implies (1). Suppose $P \not\equiv_s Q$, i.e., there exists a set R of rules, such that $\mathcal{AS}(Gr(P \cup R)) \neq \mathcal{AS}(Gr(Q \cup R))$, i.e., $\mathcal{AS}(Gr(P \cup R), U_{P\cup R}) \neq \mathcal{AS}(G(Q \cup R), U_{Q\cup R})$. Take now $C = U_{P\cup Q\cup R}$. By Lemma 2, we get $\mathcal{AS}(Gr(P \cup R, C)) \neq \mathcal{AS}(Gr(Q \cup R, C))$, and furthermore we obtain $\mathcal{AS}(Gr(P,C) \cup Gr(R,C)) \neq \mathcal{AS}(Gr(Q,C) \cup Gr(R,C))$. By results on strong equivalence in the propositional case, we get $SE(Gr(P,C)) \neq SE(Gr(Q,C))$, thus (2) does not hold.
>
> (3) implies (2). Let $C \subseteq \mathcal{C}$ such that $SE(Gr(P,C)) \neq SE(Gr(Q,C))$. By Lemma 3, this implies $SE(Gr(P,\mathcal{C})) \neq SE(Gr(Q,\mathcal{C}))$. We show that $SE(Gr(P,\mathcal{C})) \neq SE(Gr(Q,\mathcal{C}))$ implies $SE(Gr(P,D)) \neq SE(Gr(Q,D))$ with $D = U^+_{P\cup Q}$.
>
> Without loss of generalization, suppose that some $(J,I) \in SE(Gr(P,\mathcal{C}))$, such that $(J,I) \notin SE(Gr(Q,\mathcal{C}))$. From the latter, we get that there exists an $r \in Gr(Q,C)$ such that $J \not\models r^I$ (again, this holds also for $J = I$). Consider now $C' = \mathcal{C}_{P\cup Q} \cup \mathcal{C}_{\{r\}}$, $I' = (I \cap B_{\mathcal{A},C'})$, and $J' = (J \cap B_{\mathcal{A},C'})$. We have $r \in Gr(Q,C')$ and $r^{I'} = r^I$. Thus $J' \not\models r^{I'}$, and hence $(J',I') \notin SE(Gr(Q,C'))$. On the other hand, we derive $(J',I') \in SE(Gr(P,C'))$

by Lemma 3. Hence, $SE(Gr(P, C')) \neq SE(Gr(Q, C'))$. Suppose now any bijective mapping from the constants in C' which are not in $U_{P \cup Q}$ to the constants from $U^+_{P \cup Q}$ not in $U_{P \cup Q}$. Note that $U^+_{P \cup Q}$ is big enough for this, since it has additional constants for any variable in a rule. It is easily checked that such a mapping shows $SE(Gr(P, D)) \neq SE(Gr(Q, D))$.

(1) implies (3). Suppose $SE(Gr(P, D)) \neq SE(Gr(Q, D))$ and without loss of generalization, let some $(J, I) \in SE(Gr(P, D))$ satisfy $(J, I) \notin SE(Gr(Q, D))$; (the other case is by essentially the same argumentation). By the known result on strong equivalence for the propositional case, we get that there exists a ground program R, such that $\mathcal{AS}(Gr(P, D) \cup R) \neq \mathcal{AS}(Gr(Q, D) \cup R)$. Since R is ground, $\mathcal{AS}(Gr(P, D) \cup R) = \mathcal{AS}(Gr(P \cup R, D))$ and $\mathcal{AS}(Gr(Q, D) \cup R) = \mathcal{AS}(Gr(Q \cup R, D))$, and therefore $\mathcal{AS}(Gr(P \cup R, D)) \neq \mathcal{AS}(Gr(Q \cup R, D))$. Moreover, from the propositional setting we know that we can assume R to be given over all ground atoms from $Gr(P \cup Q, D)$. Hence, in particular $\mathcal{C}_R \subseteq D$. Finally, we add, for each $d \in D$, dummy facts $p(d)$, with p a fresh predicate, to R; call this extension R'. We still have $\mathcal{AS}(Gr(P \cup R', D)) \neq \mathcal{AS}(Gr(Q \cup R', D))$. Now $D = U_{P \cup R'} = U_{Q \cup R'}$ yielding by definition of answer sets for non-ground programs, $\mathcal{AS}(P \cup R') \neq \mathcal{AS}(Q \cup R')$, and whence, $P \not\equiv_s Q$.