Axiom of Choice, Maximal Independent Sets, Argumentation and Dialogue Games

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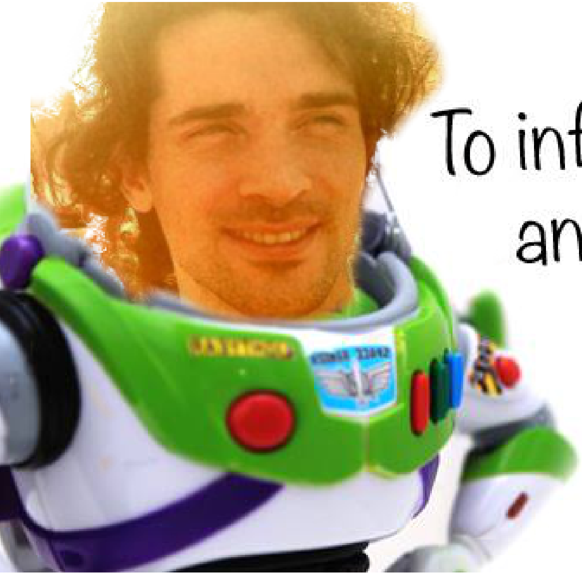


To infinity
and beyond...

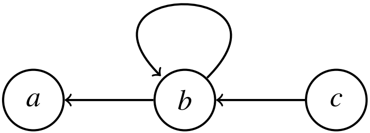
Outline

1. Introduction
   - Games and Motivations
   - Infinity and Questions
2. Backgrounds
   - Zermelo-Fraenkel Set Theory and related Axioms
   - Games Again, Infinite Style
3. The Stuff
   - Abstract Argumentation
   - An Equivalence Proof

A minor example

Example (Games played on Argument Graphs)

- Can you defend an argument a beyond doubt, i.e. defeat any attackers without running into conflict with your own argument base?
- Who has a winning strategy, you as the proponent or your oponent?



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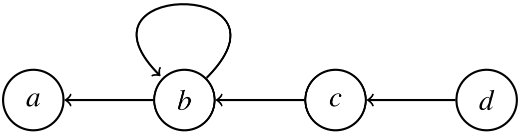

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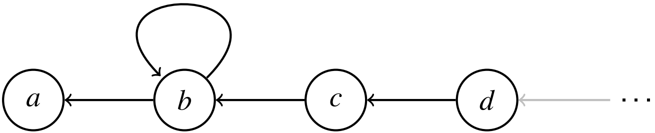

The *Why?* of Infinities I

Question

How many prime numbers are there?

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How many rational numbers $\frac{p}{q}$ *are there?*

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How many rational numbers $\frac{p}{q}$ are there?

Question

How many decimal numbers are there?

Question

Is there a set of all sets?

The *Why?* of Infinities II

Example ($|\mathbb{Q}| = |\mathbb{N}|$)

There are only as many rational as natural numbers.

$$\begin{array}{ccccc} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Example ($|\mathbb{N}| < |\mathbb{R}|$)

There are more real than natural numbers.

$$\begin{array}{llllll} i_1 = 0. & i_{1,1} & i_{1,2} & i_{1,3} & i_{1,4} & \cdots \\ i_2 = 0. & i_{2,1} & i_{2,2} & i_{2,3} & i_{2,4} & \cdots \\ i_3 = 0. & i_{3,1} & i_{3,2} & i_{3,3} & i_{3,4} & \cdots \\ i_4 = 0. & i_{4,1} & i_{4,2} & i_{4,3} & i_{4,4} & \cdots \\ \quad\vdots & & & & & \end{array}$$

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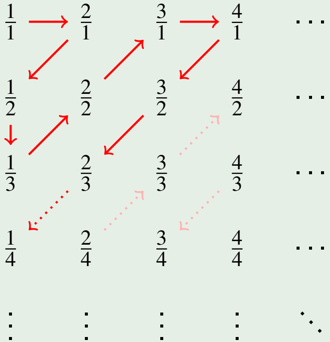

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Outline

Set Theory

Definition

Zermelo-Fraenkel Set Theory (ZFC-Axioms)

1. Extensionality $\forall x \forall y \left(\forall z \left(z \in x \Leftrightarrow z \in y \right) \Rightarrow x = y \right)$
2. Foundation $\forall x \left(\exists a \left(a \in x \right) \Rightarrow \exists y \left(y \in x \land \neg \exists z \left(z \in y \land z \in x \right) \right) \right)$
3. Specification $\forall z \forall v_1 \forall v_2 \cdots \forall v_n \exists y \forall x \left(x \in y \Leftrightarrow \left(x \in z \land \varphi \right) \right)$
4. Pairing $\forall x \forall y \exists z \left(x \in z \land y \in z \right)$
5. Union $\forall x \exists z \forall y \forall v \left(\left(v \in y \land y \in x \right) \Rightarrow v \in z \right)$
6. Replacement
   $\forall x \forall v_1 \forall v_2 \cdots \forall v_n \left(\forall y \left(y \in x \Rightarrow \exists ! z \varphi \right) \Rightarrow \exists w \forall y \left(y \in x \Rightarrow \exists ! z (y \in w \land \varphi \right) \right)$
7. Infinity $\exists x \left(\emptyset \in x \land \forall y \left(y \in x \Rightarrow \left(y \cup \{y\} \right) \in x \right) \right)$
8. Power Set $\forall x \exists y \forall z \left(z \subseteq x \Rightarrow z \in y \right)$
9. Choice $\forall x \left(\emptyset \notin x \Rightarrow \exists f : x \to \bigcup x, \forall a \in x \left(f(a) \in a \right) \right)$

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Choice and Companions

Example (The Axiom of Choice)

Every set of non-empty sets has a choice function, selecting exactly one element from each set.

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Every vector space has a basis.

Example (Well-ordering Theorem)

Every set can be well-ordered.

Example (Zorn's Lemma)

If any chain of a non-empty partially ordered set has an upper bound then there is at least one maximal element.

Determinacy

Example (A number game)

- Some well-known set of sequences of natural numbers $\mathbb{S} \subseteq \mathbb{N}^{\mathbb{N}}$, defines the winning set.
- Move i selects a number for position i, two players alternate, proponent starts with move 0.
- Proponent wins if the played sequence is an element of $\mathbb{S}$, otherwise opponent wins.

Definition (Axiom of Determinacy)

Every number game of the above form is predetermined, i.e. one of the players has a winning strategy.

Possibly infinite Games

Example (Some number game)

- Two players alternate stating moves.
- Moves are decimal digits $0, 1, \cdots 10$.
- Proponent wins if $0.i_0 i_1 i_2 i_3 \cdots \in \mathbb{Q}$.

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Example (A slightly simpler number game)

- Two players alternate making moves $i_0, i_1, i_2, i_3, \ldots$
- Moves are binary digits 0 or 1.
- The winning set (for proponent) consists of sequences where for some $n > 0$ we have $i_j = i_{j+n}$ for all $j < n$, i.e. the initial sequence is repeated at least once.
- For instance in $0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 0, 1, \cdots$ who wins?

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- For instance in $\color{blue}{0, 1, 0}, \color{red}{0, 1, 0}, 1, 1, 0, 0, 0, 1, \cdots$ proponent wins.

Choice and Determinacy

Question

How do the axioms of choice (AC) *and determinacy* (AD) *relate to each other?*

Choice vs. Determinacy

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$$(AD) \Rightarrow (AC)_{fin}$$

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Theorem (AD implies Consistency of ZF Set Theory)

$$(AD) \Rightarrow \mathit{Con(ZF)}$$

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Theorem (AD implies countable AC)

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Theorem (AD implies Consistency of ZF Set Theory)

$$(AD) \Rightarrow \mathit{Con(ZF)}$$

Theorem (AC implies not AD)

$$(AC) \Rightarrow \neg(AD)$$

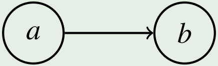
Outline

Abstract Argumentation I

Definition (Argumentation Frameworks)

- An *argumentation framework (AF)* is a pair $F = (A, R)$.
- A is an arbitrary set of *arguments*.
- $R \subseteq (A \times A)$ is the attack relation.
- For $(a, b) \in R$ write $a \rightarrowtail b$, and say a *attacks* b.
- For $a \rightarrowtail b \rightarrowtail c$ say a *defends* c against b.

Example



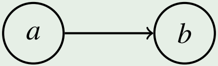
Abstract Argumentation II

Definition (Argumentation Semantics)

Some AF $F = (A, R)$ and some set $E \subseteq A$.

- E is *conflict-free (cf)* iff $E \not\rightarrowtail E$.
- E is *admissible (adm)* iff $E \in cf(F)$ and for all $a \rightarrowtail E$ also $E \rightarrowtail a$.
- E is a *preferred extension (pref)* iff it is maximal admissible, i.e. $E \in adm(F)$ and for any $E' \in adm(F)$ with $E \subseteq E'$ already $E = E'$.

Example



$cf(F) = \{\emptyset, \{a\}, \{b\}\}$ $\quad adm(F) = \{\emptyset, \{a\}\}$ $\quad prf(F) = \{\{a\}\}$

(AC) $\Rightarrow prf(F) \neq \emptyset$

Definition (Zorn's Lemma)

If any chain of a non-empty partially ordered set has an upper bound then there is at least one maximal element.

Definition (Partial Order)

A *partial order* $(P, \leq)$ is a set P with a binary relation $\leq$ that fulfills

- reflexivity: $a \leq a$,
- antisymmetry: $a \leq b \wedge b \leq a \Rightarrow a = b$,
- transitivity: $a \leq b \wedge b \leq c \Rightarrow a \leq c$.

Definition (Axiom of Union)

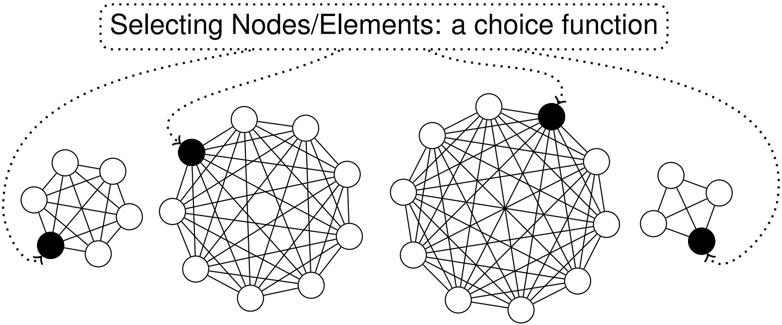
The union over the elements of a set is a set.

$$\forall z \exists y \forall x \forall u (x \in z \wedge u \in x) \Leftrightarrow u \in y$$

$(\forall F prf(F) \neq \emptyset) \Rightarrow$**(AC)**

Definition (ZF-Axioms)

- Comprehension: we can construct formalizable subsets of sets.
- Union: the union over the elements of a set is a set.
- Replacement: definable functions deliver images of sets.
- Power Set: we can construct the power set of any set.



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