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Perfection in Abstract Argumentation[1]

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Abstract. It is a well-known fact that stable semantics might not provide any extensions for some given abstract argumentation framework. Arguably such frameworks might be considered futile, at least with respect to stable semantics. We propagate σ-perfection stating that for a given argumentation graph all induced subgraphs provide σ-extensions. We discuss perfection and conditions for popular abstract argumentation semantics and possibly infinite frameworks.

Keywords. argumentation, semantics, foundations, existence, perfection

Introduction

Abstract argumentation uses arguments and a two-valued attack relation as atomic structure, and semantics to assign acceptance states to sets of arguments. In his seminal paper Dung in 1995 [1] already gave conditions for semantics to provide extensions but also examples of meaningful argumentation systems without stable extensions. Subsequently various semantics have been introduced not least to circumvent the problem of vanishing extension sets. In this work we elaborate on structural extension existence conditions. To this end we draw inspiration from kernel-perfection [2]. Given semantics σ, an argumentation framework is σ-perfect if every induced subframework provides σ-extensions. To flesh out σ-perfection in abstract argumentation we advance on known results and present novel approaches particularly for semi-stable and stage semantics.

Non-interference, contaminating frameworks and crash have been popularized as properties of argumentation semantics [3]. For various reasons these properties do not match our intuitions. When thinking about abstract argumentation semantics intuitively we want to be able to evaluate independent components of some framework independently from each other. We introduce this property as *well-definedness*. We elaborate on issues with the other properties in the Background section and use the term *collapse* from [4] to refer to our intuitive concept of crash (vanishing extension sets).

The remaining parts of this paper are organized as follows:

- In Section 1 we introduce all necessary background definitions and discuss the issue of well-definedness and collapse vs. non-interference and crash.
- In Section 2 we introduce perfection and present a fine collection of related results. This culminates in a rather sophisticated tool for stage semantics.
- In Section 3 we wrap up, relate to the literature, present a conjecture and discuss other possible future research directions.

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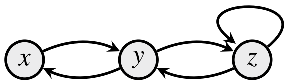


Figure 1. A simple AF as discussed in Example 1. AFs frequently are visualized as graphs where nodes reflect arguments and directed edges reflect attacks between arguments.

1. Argumentation and Fairness

Let us first introduce common definitions and basic framework operations.

Definition 1. An *argumentation framework (AF)* is an ordered pair $F = (A,R)$ where A is an arbitrary set of *arguments* and $R \subseteq A \times A$ is called the *attack* relation. For $(a,b) \in R$ we say that a attacks b. Furthermore, for $S \subseteq A$ and $a \in A$ we say that a attacks S (or S attacks a) if for some $b \in S$ we have a attacks b (or b attacks a). We use the term *defense* to denote some argument(s) attacking all attackers of some (other) argument(s). Finally, for $S \subseteq A$ we call $S^+ = S \cup \{a \in A \mid S \textit{ attacks } a\}$ the range of S in F.

For a given AF $F = (B,S)$ use $A_F = B$ and $R_F = S$ to denote its arguments and attacks respectively. For given AFs F,G with $A_F \cap A_G = \emptyset$ we use the disjunct union $F \uplus G = (A_F \cup A_G, R_F \cup R_G)$. For given AF F and argument set $X \subseteq A_F$ we use the restriction operator $F|_X = (X, X \times X \cap R_F)$.

Investigating some arbitrary AF we consider sets of arguments, and investigate whether these sets appear to be justified under some principles, also called argumentation semantics. For a comprehensive introduction into argumentation semantics see [3]. Additional to semantics discussed in [1] we consider semi-stable and stage semantics [5,6].

Definition 2. A *semantics* is a mapping from AFs to sets of arguments, where for any AF F and semantics σ we have $\sigma(F) \subseteq \wp(A_F)$. The members of $\sigma(F)$ are then called *σ-extensions* of F. By stating properties a specific extension has to fulfill, we will now define the semantics of interest for this work.

A set $S \subseteq A_F$ is called *conflict-free* (*cf*), $S \in cf(F)$ if no member attacks any other member. $S \in cf(F)$ is called *admissible* (*ad*), $S \in ad(F)$ if it defends itself against attacks from the outside. An extension $S \subseteq A_F$ is called

- *complete* (*co*), $S \in co(F)$ if $S \in cf(F)$ and S contains all arguments defended by S,
- *grounded* (*gr*), $S \in gr(F)$ if $S = \bigcap co(F)$,
- *naive* (*na*), $S \in na(F)$ if $S \in cf(F)$ and there is no $S' \in cf(F)$ with $S \subset S'$,
- *preferred* (*pr*), $S \in pr(F)$ if $S \in ad(F)$ and there is no $S' \in ad(F)$ with $S \subset S'$,
- *stage* (*sg*), $S \in sg(F)$ if $S \in cf(F)$ and there is no $S' \in cf(F)$ with $S^+ \subset S'^+$,
- *semi-stable* (*ss*), $S \in ss(F)$ if $S \in ad(F)$ and there is no $S' \in ad(F)$ with $S^+ \subset S'^+$,
- *stable* (*sb*), $S \in sb(F)$ if $S \in cf(F)$ and $S^+ = A_F$.

Example 1. Consider the AF $F = (\{x,y,z\}, \{(x,y),(y,x),(y,z),(z,y),(z,z)\})$ as depicted in Figure 1. Here the arguments could for instance refer to sentences such as x:*(everything is finite)*, y:*(infinity is real)*, z:*(reality is finite infinity)*. We have $cf(F) = ad(F) = co(F) = \{\emptyset, \{x\}, \{y\}\}$, $gr(F) = \{\emptyset\}$, $na(F) = pr(F) = \{\{x\},\{y\}\}$, $sg(F) = ss(F) = sb(F) = \{\{y\}\}$. Observe that these equality relations do not hold for arbitrary AFs. However for any AF F it holds that $sb(F) \subseteq sg(F) \subseteq na(F) \subseteq cf(F)$ and $sb(F) \subseteq ss(F) \subseteq pr(F) \subseteq ad(F) \subseteq cf(F)$.

In opposition to the traditional semantics properties of crash-resistance and non-interference we will use a different word to denote a formally different meaning.

Definition 3 (Collapse)**.** A semantics σ is said to *collapse* for some AF F if $\sigma(F) = \emptyset$.

We now give intuitive properties for semantics with the main principle of fairness in mind. There should be acceptable arguments for some frameworks. Arguments should be treated equally. We should be able to evaluate components of the union of disjunct AFs independently from each other.

Definition 4 (Fairness)**.** An argumentation semantics σ is called

1. *basic* if there is some AF F and argument set $S \neq \emptyset$ such that $S \in \sigma(F)$;
2. *language independent* [3] if isomorphic AFs produce isomorphic extension sets;
3. *well-defined* if it evaluates separate components separately, for AFs F, G, H with $H = F \uplus G$ we have $\sigma(H) = \{S \cup T \mid S \in \sigma(F), T \in \sigma(G)\}$;
4. *fair* if it is basic, language independent and well-defined.

All semantics under consideration are fair semantics. We even go a bit further and state that only fair semantics are of use for abstract argumentation. For the purpose of reference we give a formal definition of non-interference and crash-resistance and follow up by showing equivalence of collapse with crash and interference for fair semantics.

Definition 5. A semantics σ is *non-interfering* if for AFs F, G, H with $H = F \uplus G$ we have $\sigma(F) = \{S \cap A_F \mid S \in \sigma(H)\}$. A semantics σ is *crash-resistant* if there is no AF F such that for all disjunct AFs G we have $\sigma(F \uplus G) = \sigma(F)$, otherwise it *crashes* at F.[2]

Lemma 1. *A given fair semantics σ collapses for some AF F if and only if it violates crash-resistance and non-interference.*

Proof. Assume $\sigma(F) = \emptyset$ for some AF F. By well-definedness for any disjoint AF G we get $\sigma(F \uplus G) = \{S \cup T \mid S \in \emptyset, T \in \sigma(G)\} = \emptyset$, i.e. σ crashes at F and (in case $\sigma(G) \neq \emptyset$, granted σ is basic language-independent) also violates the non-interference property.

Now assume σ does not collapse for any AF and consider some arbitrary syntactically disjoint AFs F and G, and $H = F \uplus G$. Since σ does not collapse we have $\sigma(F) \neq \emptyset$ and $\sigma(G) \neq \emptyset$. By well-definedness we then get $\sigma(H) = \{S \cup T \mid S \in \sigma(F), T \in \sigma(G)\}$ and hence non-interference. With σ being basic wlog. there is some AF F with $S \in \sigma(F)$ and $S \neq \emptyset$. By definition of semantics and disjointness we get $S \cap \bigcup \sigma(G) = \emptyset$. With $\sigma(G) \neq \emptyset$ there is $T \in \sigma(G)$ and hence with $S \cup T \notin \sigma(G)$ no AF G can crash σ. □

Regarding erratic behaviour of non-interference and crash-resistance we resume by letting go of well-definedness for the brief moment of the following example. Then, e.g. non-interference does not literally prevent interference anymore. Since we firmly believe that all reasonable semantics are fair, the main benefit of collapse over interference, contamination and crash though is a substantially less complicated characterization.

Example 2. Consider a semantics σ such that for some AFs F, G, $H = F \uplus G$ we have $\sigma(F) = \{S_i \mid i \in \mathbb{N}\}$, $\sigma(G) = \{T_i \mid i \in \mathbb{N}\}$ and $\sigma(H) = \{S_1 \cup T_i, T_1 \cup S_i \mid i \in \mathbb{N}\}$. For all we know σ might be basic, language-independent, non-interfering and not crashing. However it is not well-defined and shows strong preference for the extensions S_1 and T_1.

[2]Traditionally crash-resistance is defined via contamination, which we consider redundant.

For the next section of this paper we will characterize AFs that do not collapse for some semantics. To this end we will make use of various framework or graph classes. The remainder of this section is dedicated to introducing those.

Definition 6. An AF F is called *finite* if $|A_F| < \infty$, it is called *infinite* if it is not finite. It is called *finitary* if each argument has only finitely many attackers.

Definition 7. Given some AF F. It is called

1. *bipartite* if there is partition $B \cap C = \emptyset, A_F = B \cup C$ such that for each $(x,y) \in R_F$ we have either $x \in B$ and $y \in C$ or $y \in B$ and $x \in C$;
2. *symmetric* if for any $(x,y) \in R_F$ also $(y,x) \in R_F$;
3. *loop-free* if there is no $a \in A_F$ such that $(a,a) \in R_F$;
4. *well-founded* if there exists no infinite sequence $a_0, a_1 \cdots$ such that $(a_{i+1}, a_i) \in R_F$ for all i.

Fact 1. *It is well known [1,7,8] that*

1. *for bipartite AFs semantics pr, sg, ss, sb coincide,*
2. *for symmetric AFs every cf and ad sets (and thus na and pr, sg and ss semantics) coincide,*
3. *for symmetric loop-free AFs na, pr, sg, ss, sb coincide,*
4. *for well-founded AFs gr, co, na, pr, sg, ss, sb coincide.*

2. Perfection in Abstract Argumentation

This section is the name-giving section of this paper. We start by introducing the core definition.

Definition 8. Given some semantics σ an AF F is called σ-perfect if for any induced sub-AF F' ($F' = F|_X$ for some $X \subseteq A_F$) we have $\sigma(F) \neq \emptyset$.

The following theorem might be considered basic knowledge of abstract argumentation. The mere reason we provide proof is to highlight that Zorn's Lemma is not needed here after all.

Theorem 1. *For $\sigma \in \{cf, ad, co, gr\}$ every AF is σ-perfect.*

Proof. First the empty set always is conflict-free and admissible and is thus an extension for *cf* and *ad*. Further every AF has a grounded extension, e.g. constructed via characteristic function:[3] starting with the empty set. At each induction step we select all arguments defended (and not attacked) by the before collected arguments. At limit steps we collect all arguments collected up to this limit step. For any AF F the (limited) set of arguments A_F witnesses that at some cardinality this procedure stops as eventually it will not be able to gather any more arguments. Finally since the grounded extension always is a complete extension every AF provides a complete extension. □

[3]The characteristic function takes a set of arguments as input and gives all defended and not attacked arguments as output. It is used in [1] to characterize grounded semantics.

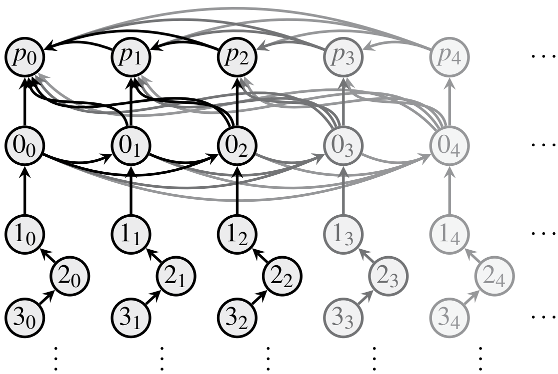


Figure 2. A cycle-free AF without stage, semi-stable or stable extensions, cf. Example 3.

Equivalence of existence of naive and preferred extensions to the Axiom of Choice is shown in [9]. For the remainder of this paper we assume ZFC and hence for instance Zorn's Lemma but do not discuss theoretical foundations thereof anymore.

Theorem 2. *Every AF is na-perfect and pr-perfect.*

We now focus on the remaining semantics *sg*, *ss* and *sb* and proceed by giving a cycle-free example of collapse.

Example 3. Consider the AF F as depicted in Figure 2. First observe that for the sequence of maximal admissible sets $S_i = \{0_i, 2_i, 4_i \cdots\} \cup \{1_j, 3_j, 5_j \cdots \mid j \neq i\}$ we have $S_i^+ \subset S_j^+$ for all $i < j$. Further observe that the p_i as well as the 0_i are pairwise in conflict and thus any conflict-free set S contains at most one of each, wlog. $p_i, 0_j \in S$. But now $S^+ \subset S^+_{\max(i,j)+1}$ and hence F collapses for *sg*, *ss* and *sb*.

It should be noted that the AF from Example 3 is cycle-free, which is why we do not overly discuss this graph-property in this paper. Now recall Fact 1 regarding basic AF classes and deduce the following.

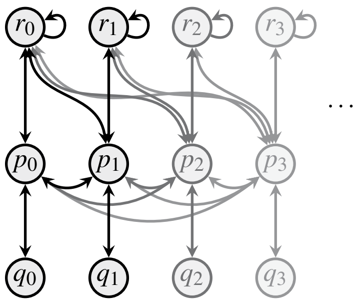
Theorem 3. *For $\sigma \in \{sg, ss, sb\}$ the following hold:*

- *bipartite AFs are σ-perfect,*
- *symmetric loop-free AFs are σ-perfect, and*
- *well-founded AFs are σ-perfect.*

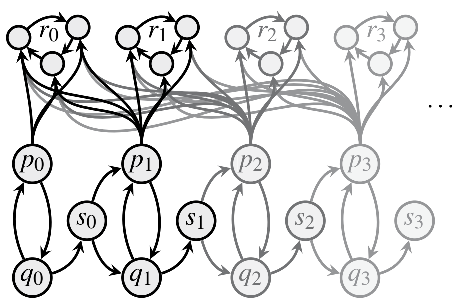
To see that neither symmetric nor loop-free AFs are σ-perfect on their own for $\sigma \in \{sg, ss, sb\}$ (and hence round out Theorem 3) we present the following two examples.

Example 4. Consider the symmetric AF F as illustrated in Figure 3(a). We have as only *pr* and *na* extensions $S = \{q_i \mid i \in \mathbb{N}\}$ and for $n \in \mathbb{N}$ the sets $S_n = (S \cup \{p_n\}) \setminus \{q_n\}$, where for $i < j$ we have $S^+ \subset S_i^+ \subset S_j^+$. So in effect for any *pr* or *na* extension there is another one of larger range and thus *sg*, *ss* and *sb* collapse.

Example 5. Consider the AF F as illustrated in Figure 3(b). The only preferred extensions are $S_q = \{q_i \mid i \in \mathbb{N}\}$ and for each $n \in \mathbb{N}$ the sets $S_n = \{q_i, p_n, s_j \mid i < n, j \geq n\}$. Here p_n



(a) Collapse in symmetric framework.



(b) Collapse in loop-free framework.

Figure 3. AFs without semi-stable or stage extensions, cf. Examples 4 and 5.

defends s_n, and accepting s_n for admissibility reasons means that we will accept each s_j for $j > n$. Again for $i < j$ we have $S_q^+ \subset S_i^+ \subset S_j^+$, and hence the collapse of semi-stable semantics. It can be shown that F collapses also for stage semantics, see [4] for a proof for a similar example.

As σ-perfection is inspired by kernel-perfection from graph theory and for any AF F the digraph $D = (A_F, \{(b,a) \mid (a,b) \in R_F\})$ has the set S as a kernel if and only if $S \in sb(F)$ we continue by importing the following two theorems.

Theorem 4 (Imported and transformed from [10]). *An AF F is sb-perfect if every induced sub-AF provides a non-empty admissible set. A finitary AF F is sb-perfect if and only if every finite induced sub-AF provides a sb extension.*

Theorem 5 (Imported and transformed from [11]). *Some given finite AF F is sb-perfect if every cycle of odd length is symmetrical.*

With this we close the case on stable semantics and move on to stage and semi-stable semantics. We start with the remark that *sb*-perfection of course implies *ss*- and *sg*-perfection and a last import.

Theorem 6 (Imported and adjusted from [12]). *Finitary AFs are sg- and ss-perfect.*

Upon our quest of searching for extensions of the given perfection-conditions for semi-stable semantics we might consider cases where the conditions are violated only marginally, for instance by one argument. The following example witnesses that this approach is of no help in the case of finitary planar[4] loop-free AFs.

Example 6. Consider the AF $F = (A, R)$ as illustrated in Figure 4. Observe that only z_0 violates the finitary condition here and that this AF is planar and loop-free.

We have as only preferred extensions the set $S_x = \{\bar{z}_0\} \cup \{x_i \mid i \in \mathbb{N}\}$ and for each $n \in \mathbb{N}$ the sets $S_n = \{x_i, y_j, \bar{z}_j \mid j \leq n, i > n\}$. Again for $i < j$ we have $S_x^+ \subset S_i^+ \subset S_j^+$ and hence semi-stable semantics collapses. For stage semantics on the other hand, the set

[4] In this paper we do not give a formal definition of an AF being planar. Informally planar AFs can be sketched on a plane without crossing attack lines.

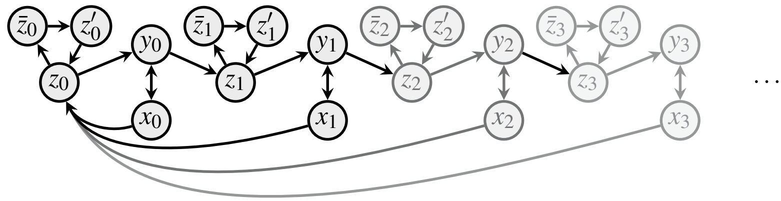


Figure 4. Loop-free planar AF with all but one finitary arguments and *ss*-collapse, cf. Example 6.

$S_y = \{y_i, \bar{z}_i \mid i \in \mathbb{N}\}$ is maximal in range, as only $z_0 \notin S_y^+$. But attacking z_0 means including z'_0 or x_j for some j and thus one of $\bar{z}_0$, z_{j+1}, $\bar{z}_{j+1}$ or z'_{j+1} drops out of range.

We now turn to stage semantics and start straightforward with a powerful result. We will then give example applications of this characterizing theorem.

Theorem 7 (Stage Perfection Characterization)**.** *Given some AF $F = (A,R)$ where there is a finite set $Y \subseteq A$ such that the restriction $F|_{A\setminus Y}$ is sg-perfect. Then also F is sg-perfect.*

Proof. We use induction on the size of Y where the base case is given by assumption. We hence assume $Y = \{x\}$ as induction step. Observe that for every naive extension $S \in na(F)$ we can distinguish three cases:

1. $x \in S$ (x is a member of S),
2. $x \in S^+ \setminus S$ (S attacks x),
3. $x \in A \setminus S^+$ (due to maximality then however x attacks S).

For a contradiction assume $\sigma(F) = \emptyset$, yet for every proper induced sub-AF $F' = F|_{A\setminus Y}$ for $x \in Y \subseteq A$ we have $\sigma(F') \neq \emptyset$. This means that there is an unbounded range-chain $(S_i)_{i\in\mathbb{N}}$ of $S_i \in na(F)$ such that for $i < j$ we have $S_i^+ \subset S_j^+$. As this range-chain clearly can not be finite there is an infinite amount of S_i that can be filed under one and the same of above three cases. We proceed by considering each of these cases separately.

Case (1), wlog. $x \in S_i$ for all i: Then for each i we have $x^+ \subseteq S_i^+$ and hence $(S_i \setminus \{x\})_i$ is an unbounded naive range-chain of $F|_{A\setminus\{x,a,b \mid (a,x),(x,b)\in R\}}$ already.[5]

Case (2), wlog. S_i attacks x for all i: Then $x \in S_i^+$ for all i and hence $(S_i)_i$ is an unbounded range-chain of $F|_{A\setminus\{x\}}$ already.

Case (3), wlog. $x \notin S_i^+$: Then clearly x is also not member of the chain-range $\bigcup_{i\in\mathbb{N}} S_i^+$ and thus $(S_i)_i$ is an unbounded range-chain for $F|_{A\setminus\{x\}}$ already again. □

The full power of Theorem 7 comes into play when considering classes of AFs we already know to be *sg*-perfect. We can immediately extend these classes and do so with the following corollaries. The first is dual to and thus proof of a conjecture from [4], i.e. *sg* collapses only if there are infinitely many arguments with infinitely many attackers. Recall that finitary AFs are *sg*-perfect.

Corollary 1. *AFs where most arguments have only finitely many attackers are sg-perfect.*

For the following recall that in symmetric AFs *cf* and *ad* and thus *sg* and *ss* coincide, and that symmetric loop-free AFs (see Theorem 3) are *sg*-perfect.

Corollary 2. *Symmetric AFs with finitely many self-attacking arguments are sg/ss-perfect.*

[5] In case of semi-stable this case is the reason the theorem fails, as $S_i \setminus \{x\}$ might not be admissible.

3. Discussion

In a way this paper is a collection of subtle details. In Section 1, Lemma 1 and Example 2 we critically discuss non-interference, contamination and crash-resistance. We proclaim (Definitions 3 and 4) well-definedness, fair semantics and collapse instead. In Section 2 we introduce and raise awareness for σ-perfection. Naturally such an intuitive property provides several results almost for free, or as corollaries from e.g. [1,7,8,9,12]. Still, especially for semi-stable and stage semantics we advance on known results and collapsing examples, proof a conjecture from [4] and elaborate on the surprisingly profound resistance of stage semantics against collapse (Theorem 7). With this we get by themselves already very powerful results (e.g. Corollaries 1 and 2) seemingly for free.

As obvious future research questions there are several other semantics out in the wild to be considered. Further results from graph theory on kernel-perfection can deliver additional immediate results for *sb*-perfection (and thus *ss*- and *sg*-perfection). It might also prove rather useful to consider classes of finitely generated infinite argumentation frameworks. Finally, also other syntactical AF-properties might be of interest in terms of σ-perfection. For instance, above results, the dynamics of chain-ranges and range-chains [12] and observations on the density of attacks in *sg*-collapsing AFs [4] let us propose this closing conjecture.

Conjecture 1. *Planar AFs are sg-perfect.*

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