# Hunt for the Collapse of Semantics in Infinite Abstract Argumentation Frameworks*

**Christof Spanring**[1,2]

1 **Department of Computer Science**[†]**, University of Liverpool, UK**
c.spanring@liverpool.ac.uk
2 **Institute of Information Systems, TU Wien, Austria**

**Abstract**

In this work we discuss examples of infinite abstract argumentation frameworks (AFs). Our focus is mainly on existence of extensions of semantics such as semi-stable and stage semantics, as opposed to the collapse where some argumentation frameworks prevent any extension. We visit known examples from the literature and present novel variants. Finally, we also give insights into extension existence conditions.

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## 1 Introduction

In everyday life we hardly ever think of dealing with actually infinite structures. Our time on earth may be complicated but it appears to be strictly finite, we deal with finite space, finite distances and finite cardinalities, i.e. natural numbers. Computer scientists in particular, tend to prefer working with finite structures, e.g. algorithms are supposed to terminate in a finite amount of time. Often enough infinity introduces odd behaviour and exceptions to the languages we use, and grew to love. For instance consider some countable language of finite words over some finite alphabet (e.g. English or C++). Think about effectively spelling an infinite word now. However, infinite structures actually are important even in our everyday life, see [16] for a fabulous overview in that matter.

In abstract argumentation, as introduced by Dung in his seminal paper [9], we break down the art of reaching consensus to abstract arguments and attacks. Due to the practical nature of argumentation most work in the literature restricts itself to the case of only finitely many arguments and attacks. Nonetheless already Dung discussed some non-finite cases and helpful definitions. For abstract argumentation in particular, several commonly used ways of instantiation naturally produce infinite structures [1, 7], and thus on the abstract level provide reason to investigate infinite frameworks.

We focus on range-based semantics and discuss conditions for existence of extensions, respectively examples where there is no extension. Section 2 can be seen as an introduction into argumentation, Section 3 presents known and novel examples, Section 4 closes with a final discussion.

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† http://cgi.csc.liv.ac.uk/~christof


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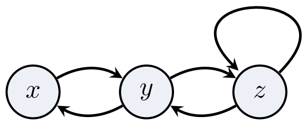


**Figure 1** A simple AF as discussed in Examples 2 and 4.

## 2 Abstract Argumentation

Abstract Argumentation was introduced by Dung in [9], motivated by philosophical works, such as [14, 19], and further on used in various fields, ranging from legal reasoning [4], to non-monotonic logic [6], artificial intelligence [5] and others.

▶ **Definition 1.** An *argumentation framework (AF)* is an ordered pair $F = (A, R)$ where $A$ is an arbitrary set of *arguments* and $R \subseteq A \times A$ is called the *attack* relation. For $(a, b) \in R$ we say that $a$ attacks $b$, for $(a, b), (b, c) \in R$ we say that $a$ defends $c$ against $b$. Furthermore, for $S \subseteq A$ and $a \in A$ we say that $a$ attacks $S$ (or $S$ attacks $a$) if for some $b \in S$ we have $a$ attacks $b$ (or $b$ attacks $a$). We extend this notion also for $S, T \subseteq A$ accordingly. Finally, for $S \subseteq A$ we call $S^+ = S \cup \{a \in A \mid S \textit{ attacks } a\}$ the range of $S$ in $F$.

AFs frequently are visualized as a graphs where nodes reflect arguments and directed edges reflect attacks between arguments.

▶ **Example 2.** Consider the AF $F = (A, R)$ depicted in Figure 1. We have $A = \{x, y, z\}$ and $R = \{(x, y), (y, x), (y, z), (z, y), (z, z)\}$. Here the arguments could for instance refer to sentences such as $x$:*(everything is finite)*, $y$:*(infinity is real)*, $z$:*(reality is finite infinity)*.

Investigating some arbitrary AF we will consider sets of arguments, and investigate whether these sets appear to be justified under some principles, also called argumentation semantics. For a comprehensive introduction into argumentation semantics see [2].

▶ **Definition 3.** An *argumentation semantics* is a mapping from AFs to sets of arguments, where for any AF $F = (A, R)$ and semantics $\sigma$ we have that if $S \in \sigma(F)$ then $S \subseteq A$. The members of $\sigma(F)$ are then called *$\sigma$-extensions* of $F$. By stating properties a specific extension has to fulfill, we will now define the semantics of interest for this work.

A set of arguments $S \subseteq A$ is called *conflict-free* ($cf$) if no member attacks any other member, i.e. $S \in cf(F)$ if for all $a, b \in S$ we have $(a, b) \notin R$. $S$ is further called *admissible* ($ad$) if it defends itself against attacks from the outside, i.e. $S \in ad(F)$ if $S \in cf(F)$ and for any $a \in A$ such that $a$ attacks $S$ we have that also $S$ attacks $a$. An extension $S \subseteq A$ is called

- *naive* ($na$), $S \in na(F)$ if $S \in cf(F)$ and there is no $S' \in cf(F)$ with $S \subsetneq S'$,
- *preferred* ($pr$), $S \in pr(F)$ if $S \in ad(F)$ and there is no $S' \in ad(F)$ with $S \subsetneq S'$,
- *stage* ($sg$), $S \in sg(F)$ if $S \in cf(F)$ and there is no $S' \in cf(F)$ with $S^+ \subsetneq S'^+$,
- *semi-stable* ($sm$), $S \in sm(F)$ if $S \in ad(F)$ and there is no $S' \in ad(F)$ with $S^+ \subsetneq S'^+$.

▶ **Example 4.** Consider the AF $F$ from Example 2. We have $cf(F) = ad(F) = \{\emptyset, \{x\}, \{y\}\}$, $na(F) = pr(F) = \{\{x\}, \{y\}\}$, $sg(F) = sm(F) = \{\{y\}\}$. Observe that these equality relations do not hold for arbitrary AFs. However, for general AFs by definition we always have $sg \subseteq na \subseteq cf$ and $sm \subseteq pr \subseteq ad \subseteq cf$.

▶ **Definition 5.** An AF $F = (A, R)$ is called *finite* if $|F| := |A| < \infty$, it is called *infinite* if it is not finite. Regardless of whether $F$ is finite or infinite it is called *finitary* [9] if each argument has only finitely many attackers, i.e. for all $a \in A$ we have $|\{b \in A \mid (b, a) \in R\}| < \infty$.

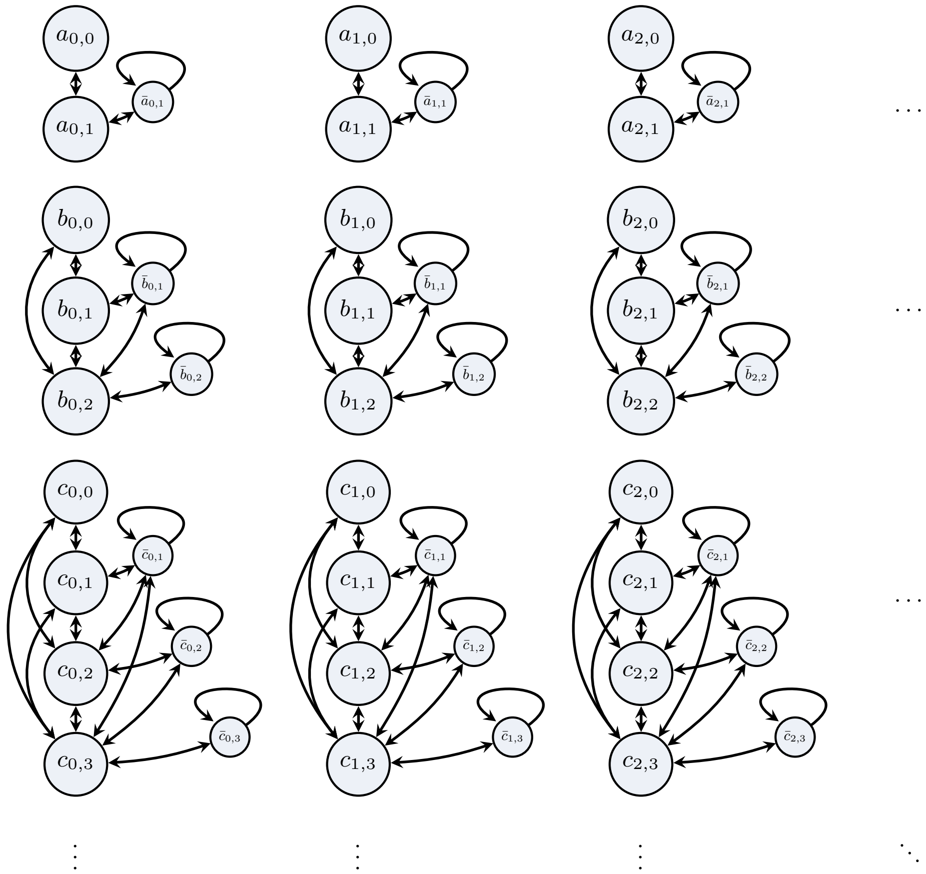

**Figure 2** Transfinitely many steps might be necessary when constructing *sg* or *sm* extensions, cf. Example 6.

For finite AFs and the given semantics it might be that we sometimes receive only empty extensions ($\sigma(F) = \{\emptyset\}$), but for finite AFs at least there will always be extensions ($\sigma(F) \neq \emptyset$). If there are infinitely many arguments similar statements are not quite as obvious.

## 3 The infinite realm

In [17] it was shown that existence of *na* or *pr* extensions for arbitrary AFs is equivalent to the axiom of choice. For this work we assume that the axiom of choice holds and thus *na* and *pr* extensions exist for arbitrary AFs. Thus further on we will focus on *sg* and *sm* semantics. For range-maximality we have that [18, 8] discusses cases where no *sg* or *sm* extension exists, and [20, 3] discuss and prove existence conditions, including finitariness. In the following we will review examples and discuss variations. Let us first take a look at difficulties we might run into with naive approaches to constructing range-maximal sets.

▶ **Example 6** (Forest of Arguments). Consider the AF $F = (A, R)$ depicted in Figure 2, where $A = \{a_{i,0}, a_{i,1}, \bar{a}_{i,1}, b_{i,0}, b_{i,1}, \bar{b}_{i,1}, b_{i,2}, \bar{b}_{i,2}, c_{i,0}, c_{i,1}, \bar{c}_{i,1}, c_{i,2}, \bar{c}_{i,2}, c_{i,3}, \bar{c}_{i,3}, \cdots \mid i \in \mathbb{N}\}$and $R = \{(x_{i,j}, x_{i,k}) \mid j \neq k\} \cup \{(x_{i,j}, \bar{x}_{i,k}), (\bar{x}_{i,k}, \bar{x}_{i,k}), (\bar{x}_{i,k}, x_{i,j}) \mid k \leq j\}$.

As in this AF all attacks are symmetric we have coinciding extension-sets of *cf* and *ad*, of *na* and *pr*, and of *sg* and *sm*. Consider the *na* set $S_0 = \{x_{i,0} \mid x \in \{a, b, c, \cdots\}, i \in \mathbb{N}\}$. If we replace $a_{0,0}$ with $a_{0,1}$ we receive $S_1 = (S_0 \cup \{a_{0,1}\}) \setminus \{a_{0,0}\}$. We further construct

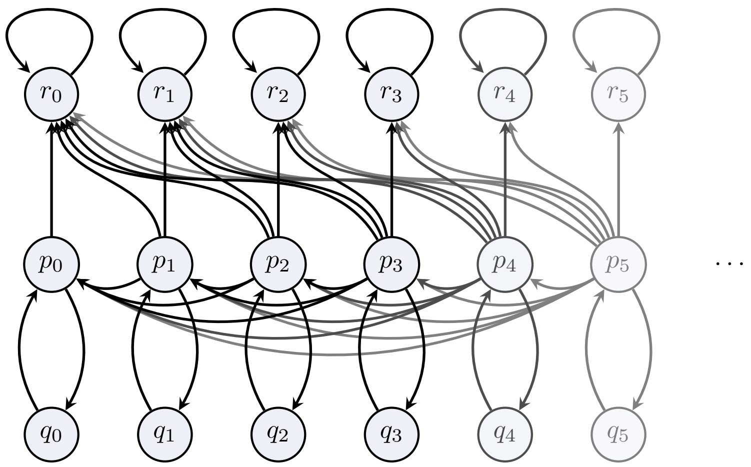


**Figure 3** A first example without semi-stable or stage extensions, cf. Example 7.

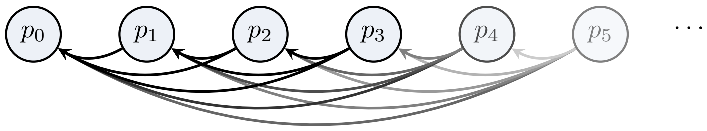


**Figure 4** Minimal AF without stage extensions, cf. Example 8.

$S_2 = (S_1 \cup \{b_{0,1}\}) \setminus \{b_{0,0}\}$, $S_3 = (S_2 \cup \{c_{0,1}\}) \setminus \{c_{0,0}\}$ and so on. We receive an infinite chain $S_0, S_1, S_2, \cdots$ of range increasing *pr* extensions $S_i$. In this case the limit $T_0 = \{a \in A \mid$ *a occurs infinitely often in* $S_i\}$ is a *pr* extension with $S_i^+ \subseteq T_0^+$ for all $S_i$. However still $T_0$ is no *sm* extension as for instance for $T_1 = (T_0 \cup \{b_{0,2}) \setminus \{b_{0,1}\}$ we have $T_0^+ \subsetneq T_1^+$.

Thus just picking random sets with greater range can result in the need of transfinitely many steps, brute-force induction might not work. Observe that the given AF is finitary, as each connected component consists of only finitely many arguments. We leave it as an exercise for the interested reader to come up with a valid *sm* extension.

## 3.1 Preliminary Examples

We now discuss examples first introduced into abstract argumentation in [18] (Examples 7 and 8). We will use the term *collapse* to refer to some semantics not providing any extension for a given AF, i.e. if $\sigma(F) = \emptyset$ we say that $\sigma$ collapses for $F$. The intuition of this wording is that existence of such AFs is problematic for modular approaches, i.e. if $F_1 = (A_1, R_1)$ and $F_2 = (A_2, R_2)$ do not share any arguments and $\sigma$ collapses for $F_1$, then $\sigma$ also collapses for $F = F_1 \cup F_2 = (A_1 \cup A_2, R_1 \cup R_2)$ regardless of possible $\sigma$-extensions for $F_2$.

▶ **Example 7.** Consider the AF $F = (A, R)$ illustrated in Figure 3 with $A = \{p_i, q_i, r_i \mid i \in \mathbb{N}\}$ and $R = \{(p_i, q_i), (q_i, p_i), (p_i, r_i), (r_i, r_i) \mid i \in \mathbb{N}\} \cup \{(p_i, p_j), (p_i, r_j) \mid j < i\}$. We have as only *pr* and *na* extensions $S = \{q_i \mid i \in \mathbb{N}\}$ and for $n \in \mathbb{N}$ the sets $S_n = (S \cup \{p_n\}) \setminus \{q_n\}$, where for $i < j$ we have $S^+ \subsetneq S_i^+ \subsetneq S_j^+$. So in effect for any *pr* or *na* extension there is another one of larger range and thus *sm* and *sg* collapse.

▶ **Example 8.** Contained as a subframework in Example 7 is the AF $F = (A, R)$, as illustrated in Figure 4, with $A = \{p_i \mid i \in \mathbb{N}\}$ and $R = \{(p_i, p_j) \mid j < i\}$. Here the only admissible set is the empty set and hence $pr(F) = sm(F) = \{\emptyset\}$. The singletons $p_i$, on the other hand, are conflict-free and even serve as naive sets. For stage semantics, however, given $S_i = \{p_i\}$

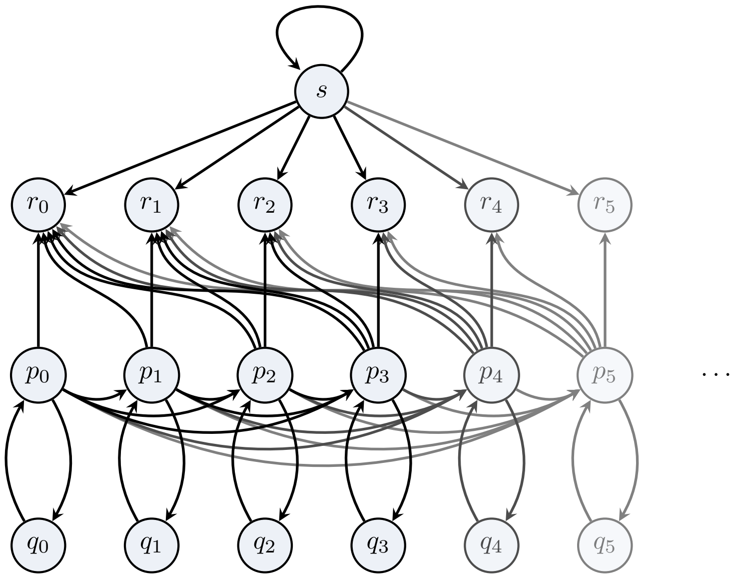


**Figure 5** Avoiding self-attacks, no semi-stable but stage extensions, cf. Example 11.

we have that for instance $S_{i+1}$ has larger range and thus *sg* collapses. So for this AF *sg* collapses but *sm* does not.

We now discuss minor modifications of Example 7 and restrictions such as Example 8.

▶ **Example 9.** Consider the AF $F = (A, R)$ from Example 7 and a symmetric version thereof $F' = (A, R')$ where $R' = R \cup \{(b, a) \mid (a, b) \in R\}$. For this AF *sg* and *sm* still collapse. However, observe that the restriction to the $p_i$, $F'_p = F'|_{\{p_i|i\in\mathbb{N}\}}$ now represents an AF where each $\{p_i\}$ is a *sg* and *sm* extension.

▶ **Example 10.** Consider the AF $F = (A, R)$ from Example 7. We now reverse the attacks between the $p_i$, $F' = (A, R')$, $R' = R \setminus \{(p_i, p_j) \mid j < i\} \cup \{(p_j, p_i) \mid j < i\}$. Again for this AF still *sm* and *sg* collapse. In other words the direction of the attacks between the $p_i$ does not matter. Observe that the restriction $F'_p = F'|_{\{p_i|i\in\mathbb{N}\}}$ now represents an AF where $\{p_0\}$ serves as sole *sg* and *sm* extension.

## 3.2 Further advancements

In this section we present novel examples and ideas. Reconsider that AFs that do not provide any stage or semi-stable extension necessarily are non-finitary, i.e. there are arguments that are attacked by an infinite amount of other arguments. In this section we approach to reduce the amount of non-finitary arguments in such AFs.

▶ **Example 11.** Consider the AF $F = (A, R)$ from Example 10, which is Example 7 with reversed attacks between the $p_i$. We replace the self-attacks as illustrated in Figure 5, $F' = (A \cup \{s\}, R')$ with $R' = R \setminus \{(r_i, r_i) \mid i \in \mathbb{N}\} \cup \{(s, s), (s, r_i) \mid i \in \mathbb{N}\}$.

For admissibility based semantics it does not matter whether some argument is self-attacking, or attacked by some other argument it can not be defended against. Thus, we still have *sm* collapsing for $F'$. However, for instance $\{r_i, q_i \mid i \in \mathbb{N}\}$ serves as stage extension.

In all the previous examples there were pairwise attacks between all $p_i$, i.e. an infinite clique of arguments. In the following two examples we will approach a maximum of letting go of the infinitary components.

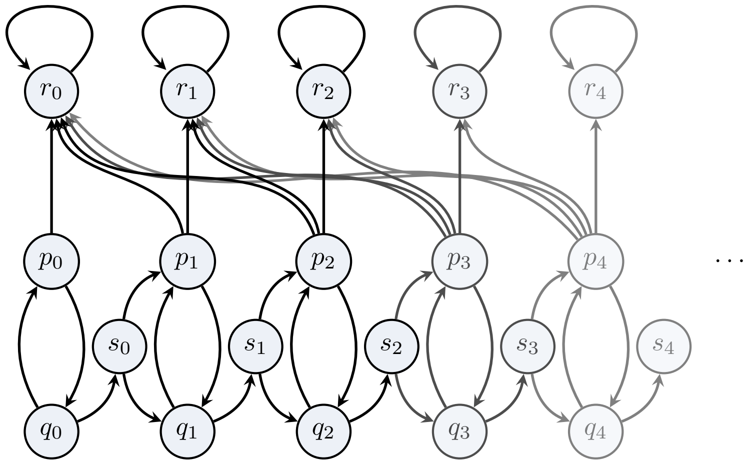


**Figure 6** Avoiding infinite cliques, no semi-stable or stage extensions, cf. Example 12.

▶ **Example 12.** Consider the AF $F = (A, R)$ from Example 7. We replace the attacks between the $p_i$ with an infinite chain of admissibility as illustrated in Figure 6, $F' = (A \cup \{s_i \mid i \in \mathbb{N}\}, R')$ where $R' = R \setminus \{(p_i, p_j)\} \cup \{(q_i, s_i), (s_i, p_{i+1}), (s_i, q_{i+1}) \mid i \in \mathbb{N}\}$.

Now observe that the only preferred extensions are $S_q = \{q_i \mid i \in \mathbb{N}\}$ and for each $n \in \mathbb{N}$ the sets $S_n = \{q_i, p_n, s_j \mid i < n, j \geq n\}$. Here $p_n$ defends $s_n$, and accepting $s_n$ for admissibility reasons means that we will accept each $s_j$ for $j > n$. Again for $i < j$ we have $S_q^+ \subsetneq S_i^+ \subsetneq S_j^+$, and hence the collapse of semi-stable semantics.

For stage semantics, on the other hand, we need to consider more candidates, as also $S_p = \{p_i \mid i \in \mathbb{N}\}$ and any feasible combination between $p_i$, $q_j$ and $s_k$ serve as naive extensions. Now take some $S \in na(F)$ as given. If there is a maximal $n \in \mathbb{N}$ with $p_i \notin S$ for $i > n$, then $S_{n+1}$ as defined above has larger range than $S$. Hence assume that for each $n \in \mathbb{N}$ there is some $i > n$ with $p_i \in S$. We conclude that for some $m \in \mathbb{N}$ we have both $s_m \notin S^+$ and one of $p_{m+1} \in S$ or $q_{m+1} \in S$. We construct $S' = \{q_j \mid j \leq m\} \cup (S \cap \{p_i, q_i, s_i \mid i > m\})$. By construction $S^+ \subsetneq S'^+$, and hence stage semantics collapses for this AF as well.

Now that we have seen a vast amount of examples illustrating how close we can get to finitariness while keeping the collapse for stage semantics, we take one step further for semi-stable semantics.

▶ **Example 13.** Consider AF $F = (A, R)$ illustrated in Figure 7, with $A = \{x_i, y_i, z_i \mid i \in \mathbb{N}\}$ and $R = \{(z_i, z_i), (z_i, y_i), (x_i, y_i), (x_i, z_0), (y_i, x_i), (y_i, z_{i+1}) \mid i \in \mathbb{N}\}$. Observe that only $z_0$ violates the finitary condition here.

We have as only preferred extensions the set $S_x = \{x_i \mid i \in \mathbb{N}\}$ and for each $n \in \mathbb{N}$ the sets $S_n = \{x_i, y_j \mid j \leq n, i > n\}$. Again for $i < j$ we have $S_x^+ \subsetneq S_i^+ \subsetneq S_j^+$ and hence semi-stable semantics collapses. For stage semantics, however, the set $S_y = \{y_i \mid i \in \mathbb{N}\}$ is maximal in range, as only $z_0 \notin S_y^+$, but attacking $z_0$ means including $x_j$ for some $j$ and thus not attacking $z_{j+1}$.

For all known examples of AFs where stage semantics is collapsing we have that there is an infinite amount of arguments with infinitely many attackers. For semi-stable semantics one such argument suffices. It appears that the collapse of stage semantics requires naive range-increasing extension chains that gradually loose arguments but keep attacking them and their range.

Further for all known *sg*-collapsing examples we have that removal of a finite amount of arguments does not affect the collapse. Further insight from techniques used in [3] appears

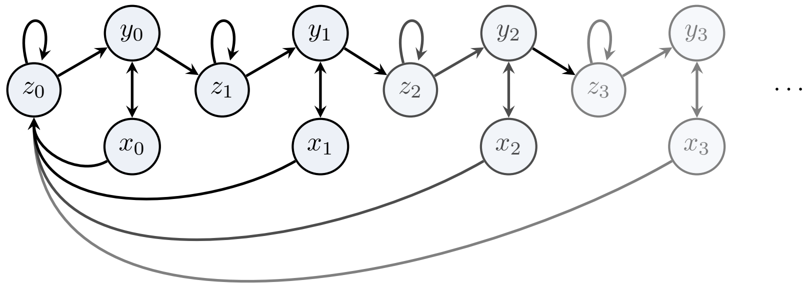


**Figure 7** Some minimal AF illustrating the collapse of semi-stable semantics, cf. Example 13.

to suggest that for *sg* this property in general holds. In more detail we claim that for any *sg* extension in any AF each member of a certain class of smaller AFs contains a corresponding extension. It is neither purpose nor objective of this paper to give a proof of this rather technical claim. However especially in the context of collapsing semantics and the idea of reducing non-finitary arguments in mind we present the following conjecture.

▶ **Conjecture 14.** *If stage semantics collapses, then there is an infinite amount of arguments with infinitely many attackers.*

As suggested technique for proving Conjecture 14 we suggest standard induction over the number of (non-finitary) arguments. Reconsider Example 13 and stage semantics. Starting with e.g. $S_0$ and clutching up the chain of the $S_i$ we receive as limit the set $S_y = \{y_i \mid i \in \mathbb{N}\}$, which is a stage extension not having $z_0$ in range. Anyhow, as $z_0 \notin S_y^+$, but $z_0 \in S_i$ for all $i \in \mathbb{N}$ we have a shift in range in the limit step. This is merely a hint on why Conjecture 14 is hard to prove.

## 4 Discussion and Future Work

We approached listing and classifying known and novel examples of AFs where stage or semi-stable semantics collapse. As the only known guarantee of existence for both semantics is an AF being finitary (see [20, 3]), the main effort was to reduce the impact of non-finitariness to a minimum. In Example 13 there is only one argument with infinitely many attackers. In this sense of minimizing non-finitary arguments, we completed the picture of collapse for semi-stable semantics, and conjectured completion for stage semantics.

Intertranslatability [12, 11] and signature [10] have shown to be valuable instruments for the investigation and comparison of (finite) AFs and semantics. In regards of infinite AFs the possible collapse of semi-stable and stage semantics is of interest on its own. On the one hand stage and stable semantics might produce the same extension-sets. On the other hand the collapse of semi-stable semantics immediately distinguishes it from preferred semantics, as opposed to the finite case. As infinite AFs have not been studied in the intertranslatability or signature context yet, also a comparison of the discussed and other semantics might still yield interesting results.

Immanent future work is further classification of conditions under which presented semantics might collapse or are ensured not to collapse. Graph-theoretical properties (other than symmetry, cf. Example 9) might be of interest. However, it seems to be more useful to investigate properties naturally induced by environments making use of argumentation. Promising candidates are any forms of instantiated argumentation, e.g. [13, 15] or [7].

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