Complexity Theory

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7. Logic-Based Abduction

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Outline

Introduction

Motivation

- Abduction is an important method in non-monotonic reasoning.
- It is mainly used in diagnosis, e.g., faulty systems, medicine
- It aims at explaining observed symptoms, e.g., malfunctions.

Reference

This part of the lecture is based on

Thomas Eiter and Georg Gottlob.
The Complexity of Logic-Based Abduction.
Journal of the ACM, 42(1):3-42 (1995).

What is abduction?

Consider the following football knowledge base.

- $weak\_defence \wedge weak\_attack \rightarrow match\_lost$,
- $match\_lost \rightarrow manager\_sad \wedge press\_angry$,
- $star\_injured \rightarrow manager\_sad \wedge press\_sad$

Suppose that we observe that the manager is sad.
What are the possible explanations in terms of *weak_defence*, *weak_attack*, and *star_injured*?

- $S_1 = \{star\_injured\}$
- $S_2 = \{weak\_defence, weak\_attack\}$
- $S_3 = \{weak\_attack, star\_injured\}$
- $S_4 = \{weak\_defence, star\_injured\}$
- $S_5 = \{weak\_defence, weak\_attack, star\_injured\}$

Formal definition of abduction

A propositional abduction problem (PAP) is a quadruple $\mathcal{P} = \langle V, H, M, T \rangle$ consisting of

- a finite set of propositional *variables* V,
- a *theory* T, which is a consistent finite set of propositional logical formulae over the variables V,
- a set of *hypotheses* $H \subseteq V$,
- and a set of *manifestations* $M \subseteq V$.

A set S satisfying $S \subseteq H$ is a solution iff

- $T \cup S$ is consistent (i.e., satisfiable)
- and $T \cup S \models M$ holds.

We denote the set of all solutions of a problem $\mathcal{P}$ by $Sol(\mathcal{P})$.

Abduction in Diagnosis

A diagnosis problem can be represented by a propositional abduction problem $\mathcal{P} = \langle V, H, M, T \rangle$ as follows:

- The theory T is the system description.
- The hypotheses $H \subseteq V$ describe the possibly faulty system components.
- The manifestations $M \subseteq V$ are the observed symptoms (describing the malfunction of the system).
- The solutions $S \in Sol(\mathcal{P})$ are the possible explanations of the malfunction.

Formal version of the football example

$$
\begin{aligned}
V = \{&\mathit{weak\_defence}, \mathit{weak\_attack}, \mathit{match\_lost},\\
&\mathit{manager\_sad}, \mathit{press\_angry}, \mathit{star\_injured}, \mathit{press\_sad}\}\\
T = \{&\mathit{weak\_defence} \land \mathit{weak\_attack} \rightarrow \mathit{match\_lost},\\
&\mathit{match\_lost} \rightarrow \mathit{manager\_sad} \land \mathit{press\_angry},\\
&\mathit{star\_injured} \rightarrow \mathit{manager\_sad} \land \mathit{press\_sad}\}\\
M = \{&\mathit{manager\_sad}\}\\
H = \{&\mathit{weak\_defence}, \mathit{weak\_attack}, \mathit{star\_injured}\}
\end{aligned}
$$

The main decision problems

Given a PAP $\mathcal{P}$ we ask:

- Solvability. $Sol(\mathcal{P}) \neq \emptyset$, i.e., does there exist a solution for $\mathcal{P}$?

Given a PAP $\mathcal{P}$ and a hypothesis $h \in H$, we ask:

- Relevance: Is h contained in at least one solution of $\mathcal{P}$?
- Necessity: Is h contained in all solutions of $\mathcal{P}$?

Remark. If h is not necessary then we also say that h is *dispensable*.

Recall the solutions of our football example:

$$\begin{aligned} Sol(\mathcal{P}) = \{&\{star\_injured\},\\ &\{weak\_defence, weak\_attack\},\\ &\{weak\_attack, star\_injured\},\\ &\{weak\_defence, star\_injured\}\\ &\{weak\_defence, weak\_attack, star\_injured\}\}\\ H = \{&weak\_defence, weak\_attack, star\_injured\} \end{aligned}$$

- Obviously, the set of solutions is non-empty.
- All three hypotheses of H are relevant.
- No hypothesis in H is necessary (i.e., they are all dispensable).

Not all solutions are of the same interest

Given two solutions, usually the “simpler” explanation is preferable. We define “simple” in one of the following ways:

- subset minimality ($\subseteq$): We only accept solutions S, s.t. no proper subset $S' \subset S$ is a solution.
- minimum cardinality ($\leq$): We only accept solutions S, s.t. there does not exist a solution S' with fewer elements, i.e., $|S'| < |S|$.
- Further preorders to eliminate undesired solutions: priorities or minimum weight (also referred to as penalties).

Recall the solutions of our football example:

$$
\begin{aligned}
Sol(\mathcal{P}) = \{ & \{star\_injured\}, \{weak\_defence, weak\_attack\}, \\
& \{weak\_attack, star\_injured\}, \\
& \{weak\_defence, star\_injured\} \\
& \{weak\_defence, weak\_attack, star\_injured\}\}
\end{aligned}
$$

- Subset minimal solutions.
  $Sol_{\subseteq}(\mathcal{P}) = \{\{star\_injured\}, \{weak\_defence, weak\_attack\}\}$
- Cardinality minimal solution.
  $Sol_{\leq}(\mathcal{P}) = \{\{star\_injured\}\}$
  Hence, *weak_defence* and *weak_attack* are no longer relevant.

Lemma

Deciding if $S \subseteq H$ *fullfills* $S \in Sol(\mathcal{P})$ *is in* $\Delta_2\mathrm{P}$ *(actually, even in* DP*).*

Proof.

By the definition, we have to check if

- $T \cup S$ is consistent (i.e., satisfiable) and
- $T \cup S \models M$ holds.

We can check if $T \cup S$ is satisfiable with an NP oracle.

The second problem is clearly in co-NP, which can be checked by yet another call to an NP oracle. In order to see the co-NP-membership, we consider the co-problem $T \cup S \not\models M$. Clearly, this problem can be decided by the following NP-algorithm: guess a truth assignment I and check that $T \cup S$ is **true** in I while M is **false** in I.

Theorem

The Solvability problem is Σ_2P-complete.

Proof.

Σ_2P-Membership. Guess a set $S \subseteq H$ and check with two calls to an NP oracle that $T \cup S$ is satisfiable and $T \cup S \models M$ holds.

Σ_2P-Hardness. Proof by reduction from the QSAT$_2$ problem. Let an arbitrary instance of the QSAT$_2$ problem be given by the formula $\varphi = (\exists X)(\forall Y)\psi(X, Y)$ with $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_l\}$. Let $X' = \{x'_1, \dots, x'_k\}$, $R = \{r_1, \dots, r_k\}$, and t be fresh variables.

Proof (continued)

Then we define an instance of Solvability as $\mathcal{P} = \langle V, H, M, T\rangle$ with

$$\begin{aligned} V &= X \cup Y \cup X' \cup R \cup \{t\} \\ H &= X \cup X' \\ M &= R \cup \{t\} \\ T &= \{\psi(X,Y) \rightarrow t\} \cup \{\neg x_i \vee \neg x_i', x_i \rightarrow r_i, x_i' \rightarrow r_i \mid 1 \leq i \leq k\} \end{aligned}$$

Obviously, this reduction is feasible in logarithmic space.

The effect of the clauses $\{\neg x_i \vee \neg x_i', x_i \rightarrow r_i, x_i' \rightarrow r_i\}$ in T is that, for every $i \in \{1, \ldots, k\}$, every solution of $\mathcal{P}$ contains exactly one of $\{x_i, x_i'\}$.

The correctness proof of the reduction is based on the following observation: For $A \subseteq X$, let A' denote the set $\{x' \mid x \in A\}$. Then every subset $A \subseteq X$ fulfills the following equivalence: For the assignment I on X with $I^{-1}(\mathbf{true}) = A$, every extension J of I to the variables Y satisfies the formula $\psi(X,Y) \Leftrightarrow A \cup (X \setminus A)'$ is a solution of $\mathcal{P}$.

Exercise. Give a full proof of the correctness of the reduction.

Theorem

The Relevance problem is Σ_2P-complete.
The Necessity problem is Π_2P-complete.

Proof of Membership.

Relevance. Guess a set $S \subseteq H$ with $h \in S$ and check with two calls to an NP oracle that $T \cup S$ is satisfiable and $T \cup S \models M$ holds.

Necessity. We show the Σ_2P-membership of the co-problem (i.e., the dispensability): Guess a set $S \subseteq H$ with $h \notin S$ and check with two calls to an NP oracle that $T \cup S$ is satisfiable and $T \cup S \models M$ holds.

Proof of Hardness

By reduction from the Solvability problem: Let an arbitrary instance of the Solvability problem be given by the PAP $\mathcal{P} = \langle V, H, M, T \rangle$. W.l.o.g., let T consist of a single formula φ and let h, h', m' be fresh variables. Then we define an instance of the Relevance (resp. the Necessity) problem with the following PAP $\mathcal{P}' = \langle V', H', M', T' \rangle$:

$$\begin{aligned} V' &= V \cup \{h, h', m'\} \\ H' &= H \cup \{h, h'\} \\ M' &= M \cup \{m'\} \\ T' &= \{\neg h \vee \varphi\} \cup \{h' \rightarrow m \mid m \in M\} \cup \{\neg h \vee \neg h', h \rightarrow m', h' \rightarrow m'\} \end{aligned}$$

Obviously, this reduction is feasible in logarithmic space and the new theory T' is consistent. We claim that the PAP $\mathcal{P}$ has at least one solution iff h is relevant in $\mathcal{P}'$ iff h' is not necessary in $\mathcal{P}'$.

Proof of Hardness (continued)

Proof idea for the correctness of this reduction.

- The clauses $\{\neg h \vee \neg h', h \rightarrow m', h' \rightarrow m'\}$ in T' have the effect that every solution of $\mathcal{P}'$ contains exactly one of $\{h, h'\}$.
- The solutions of $\mathcal{P}$ extended by h are clearly solutions of $\mathcal{P}'$.
- The effect of the disjunct $\neg h$ in the clause $\neg h \vee \varphi$ is that any subset $A \subseteq H$ is consistent with T' (simply set h to **false**).

Thus, $Sol(\mathcal{P}') = \{\{h\} \cup S \mid S \in Sol(\mathcal{P})\} \cup \{\{h'\} \cup A \mid A \subseteq H\}$ holds.

We conclude that

- h is relevant for $\mathcal{P}'$ iff $Sol(\mathcal{P}) \neq \emptyset$ holds and
- h' is dispensable (i.e., not necessary) for $\mathcal{P}'$ iff $Sol(\mathcal{P}) \neq \emptyset$ holds.

Exercise. Give a full proof of the correctness of the reduction.

Subset Minimality

Motivation

The subset minimality criterion removes redundancy.

Does there exist a subset minimal solution?

If a PAP has a solution then it also has a (subset) minimal solution.

$$Sol(\mathcal{P}) \neq \emptyset \Leftrightarrow Sol_{\subseteq}(\mathcal{P}) \neq \emptyset$$

So we are only interested in $\subseteq$- Relevance (resp. $\subseteq$-Necessity), i.e.: Is a given hypothesis contained in some (resp. all) subset minimal solutions?

Checking subset minimality

Lemma

Let $\mathcal{P} = \langle V, H, M, T \rangle$ be a PAP.
For every $S \subseteq H$, the following equivalence holds:
$S \in Sol_{\subseteq}(\mathcal{P}) \Leftrightarrow S \in Sol(\mathcal{P})$ *and for all* $h \in S$, $(S \setminus \{h\}) \notin Sol(\mathcal{P})$

Proof

$\subseteq$: $Sol_{\subseteq}(\mathcal{P}) = \{S \in Sol(\mathcal{P}) \mid \text{for all } S' \subset S,\ S' \notin Sol(\mathcal{P})\}$. Hence, any subset minimal solution S satisfies $S \setminus \{h\} \notin Sol(\mathcal{P})$ for all $h \in S$.

$\supseteq$: Let $S \in Sol(\mathcal{P})$ and $S \notin Sol_{\subseteq}(\mathcal{P})$. Hence, there exists $S' \subset S$ with $S' \in Sol(\mathcal{P})$. But then there exists $h \in S$ with $S' \subseteq S \setminus \{h\} \subset S$.
We claim that then also $S \setminus \{h\}$ is a solution of $\mathcal{P}$:

$T \cup S$ is consistent. Hence also $T \cup (S \setminus \{h\})$ is consistent.
$T \cup S' \models M$. Hence, by the monotonicity of $\models$, also $T \cup (S \setminus \{h\}) \models M$.

Complexity Results

Theorem

The $\subseteq$-Relevance problem is Σ_2P-complete.
The $\subseteq$-Necessity problem is Π_2P-complete.

Proof of Membership.

$\subseteq$-Relevance. Guess a set $S \subseteq H$ with $h \in S$ and check with two calls to an NP oracle that $T \cup S$ is satisfiable and $T \cup S \models M$ holds. Finally, check that S is a *minimal* solution.

By the previous lemma, we thus need polynomially many further calls to an NP oracle to check that $T \cup (S \setminus \{h'\}) \not\models M$ holds for every $h' \in S$.

Proof of Membership (continued).

$\subseteq$-Necessity. We show the Σ_2P-membership of the co-problem: Guess a set $S \subseteq H$ with $h \notin S$ and check with two calls to an NP oracle that $T \cup S$ is satisfiable and $T \cup S \models M$ holds. From this we may conclude that h is not necessary.

Clearly, there exists some $S' \subseteq S$, s.t. S' is a minimal solution. Moreover, since $h \notin S$, also $h \notin S'$ holds.

Proof of Hardness

By reduction from the Solvability problem: Let an arbitrary instance of the Solvability problem be given by the PAP $\mathcal{P} = \langle V, H, M, T \rangle$ with $H = \{h_1, \ldots, h_\ell\}$. W.l.o.g., let T consist of a single formula φ and let $\{h_0, h'_0, h'_1, \ldots, h'_\ell, m'_0, m'_1, \ldots, m'_\ell\}$ be fresh variables. Then we define an instance of the $\subseteq$-Relevance (resp. the $\subseteq$-Necessity) problem with the following PAP $\mathcal{P}' = \langle V', H', M', T' \rangle$:

$$
\begin{aligned}
V' &= V \cup \{h_0, h'_0, h'_1, \ldots, h'_\ell, m'_0, m'_1, \ldots, m'_\ell\} \\
H' &= H \cup \{h_0, h'_0, h'_1, \ldots, h'_\ell\} \\
M' &= M \cup \{m'_0, m'_1, \ldots, m'_\ell\} \\
T' &= \{\neg h_0 \vee \varphi\} \cup \{h'_0 \rightarrow m \mid m \in M\} \cup \\
&\quad \cup \{\neg h_i \vee \neg h'_i, h_i \rightarrow m'_i, h'_i \rightarrow m'_i \mid 0 \leq i \leq \ell\}
\end{aligned}
$$

Obviously, this reduction is feasible in logarithmic space and the new theory T' is consistent. We claim that the PAP $\mathcal{P}$ has at least one solution iff h_0 is $\subseteq$-relevant in $\mathcal{P}'$ iff h'_0 is not $\subseteq$-necessary in $\mathcal{P}'$.

Proof of Hardness (continued)

Proof idea for the correctness of this reduction.

- The hypotheses h_0, h'_0 play exactly the role of h, h' in the hardness proof of Relevance and Necessity (without $\subseteq$-minimality).
- For every $i \in \{0, \ldots, \ell\}$, the clauses $\{\neg h_i \vee \neg h'_i, h_i \rightarrow m'_i, h'_i \rightarrow m'_i\}$ in T' have the effect that every solution of $\mathcal{P}'$ contains exactly one of $\{h_i, h'_i\}$. In other words, every solution of $\mathcal{P}'$ is $\subseteq$-minimal.

For every $A \subseteq H$, we write A' to denote $\{h' \mid h \in A\}$. In summary, we thus have: $Sol_{\subseteq}(\mathcal{P}') = Sol(\mathcal{P}') = \{\{h_0\} \cup S \cup (H \setminus S)' \mid S \in Sol(\mathcal{P})\} \cup \{\{h'_0\} \cup A \cup (H \setminus A)' \mid A \subseteq H\}$.

We conclude that

- h_0 is $\subseteq$-relevant for $\mathcal{P}'$ iff $Sol(\mathcal{P}) \neq \emptyset$ holds and
- h'_0 is $\subseteq$-necessary for $\mathcal{P}'$ iff $Sol(\mathcal{P}) = \emptyset$ holds.

Minimum cardinality

Motivation

The subset minimal solutions may have different cardinalities. If the failure of any component in a system is independent of the failure of the other components, then explanations with minimum cardinality are the ones with highest probability. Moreover, the cost of repair is normally lower if fewer components have to be exchanged. Hence, explanations with minimum cardinality are preferable.

Does there exist a solution with minimum cardinality?

If a PAP has a solution then it also has a solution with minimum cardinality, i.e.: $Sol(\mathcal{P}) \neq \emptyset \Leftrightarrow Sol_{\leq}(\mathcal{P}) \neq \emptyset$.

So we are only interested in $\leq$- Relevance (resp. $\leq$-Necessity), i.e.: Is a given hypothesis in some (resp. all) solutions with minimum cardinality?

Complexity Results

Theorem

Both, the $\leq$-Relevance problem and the $\leq$-Necessity problem are $\Delta_3 \mathrm{P}[\log n]$-complete.

Proof of Membership.

Let an instance of $\leq$-Relevance (resp. $\leq$-Necessity) be given by a PAP $\mathcal{P}$ and a hypothesis h.

- Determine the minimum cardinality K of the solutions of the PAP $\mathcal{P}$. This can be done by $\log m$ (with $m = |H|$) calls to a Σ_2P-oracle asking questions of the sort "Does $\mathcal{P}$ have a solution of size $\leq k$?"
- We need one more call to a Σ_2P-oracle asking if there exists a solution S of size K, s.t. $h \in S$ (resp. $h \notin S$).

Preparation of the hardness proof

Recall that the following problem is Δ_2P[log n]-complete.

CARD-MINIMAL MODEL SAT

INSTANCE: Boolean formula φ and an atom z.
QUESTION: Is z true in a cardinality-minimal model of φ?

This problem can be generalized to an arbitrary level of the polynomial hierarchy. For every $i \geq 1$, it can be shown that the following problem is Δ_{i+1}P[log n]-complete.

CARD-MINIMAL MODEL QSAT$_i$

INSTANCE: Quantified Boolean formula
$\varphi(X) = \forall Y_1 \exists Y_2 \cdots Q Y_{i-1} \psi(X, Y_1, \ldots, Y_{i-1})$ and an atom $z \in X$.
QUESTION: Is z true in a cardinality-minimal model of $\varphi(X)$?

Proof of Hardness.

Let an arbitrary instance of the **CARD-MINIMAL MODEL QSAT**$_2$ problem be given by the formula $\varphi(X) = (\forall Y)\psi(X, Y)$ with $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_l\}$ and let $x_j \in X$ denote the distinguished atom. Let $X' = \{x'_1, \ldots, x'_k\}$, $X'' = \{x''_1, \ldots, x''_k\}$, $Q = \{q_1, \ldots, q_k\}$, $R = \{r_1, \ldots, r_k\}$, and t be fresh variables.

Then we define an instance of $\leq$-Relevance (resp. co-$\leq$-Necessity) via the following PAP $\mathcal{P} = \langle V, H, M, T\rangle$ and the hypothesis x_j (resp. x'_j):

$$
\begin{aligned}
V &= X \cup X' \cup X'' \cup Q \cup R \cup Y \cup \{t\} \\
H &= X \cup X' \cup X'' \\
M &= Q \cup R \cup \{t\} \\
T &= \{\psi(X, Y) \rightarrow t\} \cup \{\neg x_i \vee \neg x'_i, x_i \rightarrow q_i, x'_i \rightarrow q_i \mid 1 \leq i \leq k\} \cup \\
&\quad\; \{\neg x'_i \vee \neg x''_i, x'_i \rightarrow r_i, x''_i \rightarrow r_i \mid 1 \leq i \leq k\}.
\end{aligned}
$$

Obviously, this reduction is feasible in logarithmic space.

Proof of Hardness (continued).

For every $i \in \{1, \dots, k\}$, the clauses $\neg x_i \vee \neg x_i', x_i \rightarrow q_i, x_i' \rightarrow q_i$ in T make sure that every solution S of $\mathcal{P}$ contains exactly one of $\{x_i, x_i'\}$. Likewise, the clauses $\neg x_i' \vee \neg x_i'', x_i' \rightarrow r_i, x_i'' \rightarrow r_i$ in T make sure that every solution S of $\mathcal{P}$ contains exactly one of $\{x_i', x_i''\}$.

For $A \subseteq X$, let A' denote the set $\{x' \mid x \in A\}$ and let A'' denote the set $\{x'' \mid x \in X\}$. For every subset $A \subseteq X$, the following equivalences hold:

The assignment I on X with $I^{-1}(\mathbf{true}) = A$ is a model of $\varphi(X) \Leftrightarrow$ $A \cup (X \setminus A)' \cup A''$ is a solution of $\mathcal{P}$.

Moreover, I with $I^{-1}(\mathbf{true}) = A$ is a *cardinality-minimal* model of $\varphi(X)$ $\Leftrightarrow$ $A \cup (X \setminus A)' \cup A''$ is a *cardinality-minimal* solution of $\mathcal{P}$.

Thus, x_j is contained in some cardinality-minimal model of $\varphi(X) \Leftrightarrow$ x_j is $\leq$-relevant for $\mathcal{P} \Leftrightarrow x_j'$ is not $\leq$-necessary for $\mathcal{P}$.

Prioritization

Motivation

The set of hypotheses is partitioned into groups of different priorities

- which may possibly express a kind of probability when no numerical values are available
- and which may be used to refine other minimality criteria ($\subseteq$, $\leq$)

Definition

Prioritization on $\subseteq$. Let H be the set of hypotheses and $P = \langle P_1, \ldots, P_k \rangle$ such that $P_1 \cup \cdots \cup P_k = H$ and $P_i \cap P_j = \emptyset$ for all $i \neq j$.
We define the relation $\subseteq_P$ as follows:
$A \subseteq_P B \Leftrightarrow A = B$ or there exists $i \in \{1, \ldots, k\}$ such that:

$$A \cap P_j = B \cap P_j \text{ for } 1 \leq j < i \text{ and } A \cap P_i \subset B \cap P_i$$

Prioritization

Example

(the football example revisited)

$$\begin{aligned} V &= \{weak\_defence, weak\_attack, match\_lost, \\ &\quad manager\_sad, press\_angry, star\_injured, press\_sad\} \\ T &= \{\ldots\} \\ M &= \{manager\_sad\} \\ H &= \{weak\_defence, weak\_attack, star\_injured\} \end{aligned}$$

with prioritization: $P = \langle\{weak\_defence, weak\_attack\}, \{star\_injured\}\rangle$

$$\begin{aligned} \{weak\_defence, star\_injured\} &\subseteq_P \{weak\_defence, weak\_attack\} \\ \{weak\_defence, star\_injured\} &\not\subseteq_P \{weak\_attack\} \\ \{star\_injured\} &\subseteq_P \{weak\_defence\} \\ \{weak\_defence\} &\not\subseteq_P \{weak\_attack\} \end{aligned}$$

Complexity Results

Theorem

The $\subseteq_P$-Relevance problem is Σ_3P-complete.
The $\subseteq_P$-Necessity problem is Π_3P-complete.
These hardness results already hold for 2 priorities.

Proof of Membership.

$\subseteq_P$-Relevance. Guess a set $S \subseteq H$ with $h \in S$ and check with a call to a Δ_2P oracle that S is a solution. Moreover, we check by a call to a Σ_2P oracle that there exists no solution $S' \neq S$ with $S' \subseteq_P S$.

$\subseteq_P$-Necessity. We show the Σ_3P-membership of the co-problem: Guess a set $S \subseteq H$ with $h \notin S$ and check with a call to a Δ_2P oracle that S is a solution. Moreover, we check by a call to a Σ_2P oracle that there exists no solution $S' \neq S$ with $S' \subseteq_P S$.

Intuition of the Hardness

"$\subseteq$" Recall that subset minimality makes things only polynomially harder. This is due to the following equivalence: S is a $\subseteq$-minimal solution of the PAP $\mathcal{P}$ if and only if S is a solution of $\mathcal{P}$ and, for every $h \in S$, $S \setminus \{h\}$ is not a solution.

"$\subseteq_P$" Suppose that H is partitioned into 2 priority levels $P = \langle P_1, P_2 \rangle$. Then the following effect may occur. Suppose that S is a solution of the PAP and that, for every $h \in S$, $S \setminus \{h\}$ is not a solution. Nevertheless, it might well happen that, for some $h \in S \cap P_1$ and some $X \subseteq P_2$, the set $S' = (S \setminus \{h\}) \cup X$ is a solution. In this case, $S' \subseteq_P S$ clearly holds. Checking if such a set S' (and, in particular, if such a set X) exists comes down to yet another non-deterministic guess.

Further preorders on the solutions

Definition

Weight-minimality (Penalization). Let a PAP $\mathcal{P} = \langle V, H, M, T \rangle$ be given with a weight function w on the hypotheses. For two subsets $A \subseteq H$ and $B \subseteq H$, we write $A \leq_w B$ if $\sum_{h \in A} w(h) \leq \sum_{h \in B} w(h)$ holds.

Definition

Prioritization on $\leq$. Let H be the set of hypotheses and $P = \langle P_1, \ldots, P_k \rangle$ such that $P_1 \cup \cdots \cup P_k = H$ and $P_i \cap P_j = \emptyset, i \neq j$.
We define the relation $\leq_P$ as follows:
$A \leq_P B \Leftrightarrow A = B$ or there exists $i \in \{1, \ldots, k\}$ such that:

$$|A \cap P_j| = |B \cap P_j| \text{ for } 1 \leq j < i \text{ and } |A \cap P_i| < |B \cap P_i|$$

Complexity Results

Theorem

All of the following four problems are Δ_3P-complete:
$\leq_w$-Relevance, $\leq_w$-Necessity, $\leq_P$-Relevance, $\leq_P$-Necessity.

Proof idea.

The hardness in case of $\leq_w$ can be proved by reduction from a quantified version of **WEIGHT-MINIMAL MODEL SAT**, analogously to the proof for $\leq$. Similarly, the hardness in case of $\leq_P$ is shown by reduction from a quantified version of **LEX-MINIMAL MODEL SAT**. Below, we only sketch the proof idea of the membership.

$\leq_w$-Relevance and $\leq_w$-Necessity. First determine the minimum weight W of the solutions of $\mathcal{P}$. This can be done by a binary search asking questions like "Does $\mathcal{P}$ have a solution of weight $\leq w$?"

For this task, we need logarithmically many calls (w.r.t. the total weight) to a Σ_2P-oracle. These are polynomially many calls w.r.t. the representation of the weights of the elements in H.

Proof idea (continued)

$\leq_P$-Relevance and $\leq_P$-Necessity. Suppose that $\mathcal{P}$ has ℓ priorities and that each P_i contains n_i hypotheses. Moreover, let $|H| = \sum_{i=1}^{\ell} n_i = n$. Then we have to determine the lexicographically minimal vector $(K_1, \ldots, K_\ell)$, s.t. there exists a solution S with $|S \cap P_i| = K_i$ for each i.

This can be done in ℓ stages. In the i-th stage, we determine K_i by $\log n_i$ calls to a Σ_2P-oracle, asking questions like "Does $\mathcal{P}$ have a solution S, s.t. $|S \cap P_j| = K_j$ for all $j < i$ and $|S \cap P_i| \leq k$?"

We thus need $\mathrm{O}(\ell \cdot \log n)$ Σ_2P-oracle calls, i.e., polynomially many.

Remark. If the number of priorities is bounded by a constant, then logarithmically many Σ_2P-oracle calls suffice.

Summary

Table: Complexity results presented in the lecture

	$=$	$\subseteq$	$\subseteq_P$	$\leq$	$\leq_w$	$\leq_P$
Solvability	Σ_2P	–	–	–	–	–
Relevance	Σ_2P	Σ_2P	Σ_3P	Δ_3P[log n]	Δ_3P	Δ_3P
Necessity	Π_2P	Π_2P	Π_3P	Δ_3P[log n]	Δ_3P	Δ_3P

Summary

Table: Further complexity results in (Eiter/Gottlob, 1995)

Relevance	$=$	$\subseteq$	$\subseteq_P$	$\leq$	$\leq_w$	$\leq_P$
Horn	NP	NP	Σ_2P	Δ_2P[log n]	Δ_2P	Δ_2P
definite Horn	P	NP	NP	Δ_2P[log n]	Δ_2P	Δ_2P

Necessity	$=$	$\subseteq$	$\subseteq_P$	$\leq$	$\leq_w$	$\leq_P$
Horn	co-NP	co-NP	Π_2P	Δ_2P[log n]	Δ_2P	Δ_2P
definite Horn	P	co-NP	co-NP	Δ_2P[log n]	Δ_2P	Δ_2P

Learning Objectives

- Definition and intuition of logic-based abduction
- The main decision problems of logic-based abduction: solvability, relevance, necessity.
- Restricting the set of acceptable solutions: $\subseteq$, $\leq$, $\subseteq_P$, $\leq_w$, $\leq_P$
- Intuition why non-monotonic reasoning usually has an additional source of complexity compared with monotonic reasoning.
- Get practice with complexity results in the polynomical hierarchy.