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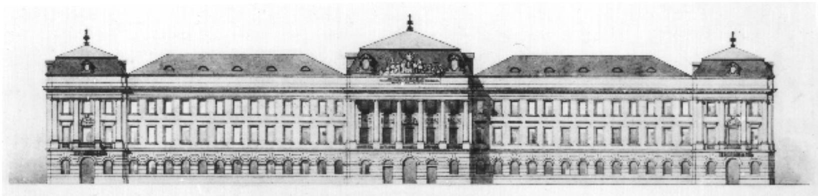

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On the Expressive Power of Collective Attacks

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On the Expressive Power of Collective Attacks

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Abstract. In this paper, we consider SETAFs due to Nielsen and Parsons, an extension of Dung's abstract argumentation frameworks that allow for collective attacks. We first provide a comprehensive analysis of the expressiveness of SETAFs under conflict-free, naive, stable, complete, admissible and preferred semantics. Our analysis shows that SETAFs are strictly more expressive than Dung AFs. Towards a uniform characterization of SETAFs and Dung AFs we provide general results on expressiveness which take the maximum degree of the collective attacks into account. Our results show that, for each $k > 0$, SETAFs that allow for collective attacks of $k + 1$ arguments are more expressive than SETAFs that only allow for collective attacks of at most k arguments.

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1 Introduction

Abstract argumentation frameworks (AFs) as introduced by Dung in his seminal paper [2] are a core formalism in formal argumentation and have been extensively studied in the literature. A popular line of research investigates extensions of Dung AFs that allow for a richer syntax (see, e.g. [1]). In this work we consider SETAFs as introduced by Nielsen and Parsons [5] which generalize the binary attacks in Dung AFs to collective attacks such that a set of arguments B attacks another argument a but no subset of B attacks a. The semantics as proposed in [5], make SETAFs a conservative generalization of Dung AFs in the sense that a SETAF that has only simple attacks is evaluated the same way as the corresponding Dung AF.

As illustrated in [5], there are several scenarios where arguments interact and can constitute an attack on another argument only if these arguments are jointly taken into account. Representing such a situation in Dung AFs often require additional artificial arguments that "encodes" the conjunction of arguments. This is also observed in a recent comprehensive investigation on translations between different abstract argumentation formalisms [6]. There, it is shown that SETAFs allow for more straightforward and compact encodings of support between arguments than AFs do. However, to the best of our knowledge, there has not been a thorough investigation to which extent the concept of collective attacks increases the expressiveness of SETAFs compared to Dung AFs.

Characterizations and comparisons of the expressiveness of argumentation formalisms (and non-monotonic formalisms in general) have been identified as a fundamental basis in order to understand the different capabilities of formalisms [3, 8, 9]. A successful notion to compare the expressiveness of argumentation formalisms is the notion of the signature [3] of a formalisms w.r.t. a semantics, that is the collection of all sets of extensions that can be expressed with at least one argumentation framework. There exist exact characterizations for most of the semantics for Dung AFs [3] and Abstract Dialectical Frameworks (ADFs) [7, 8, 9]. As already observed by Polberg [6] collective attacks allow to enforce certain sets of extensions that cannot be obtained with Dung AFs. However, there are no characterizations of the signatures for SETAFs and thus the precise differences in expressiveness to Dung AFs and ADFs are still unclear. In this work we investigate the signatures of SETAFs for conflict-free, naive, stable, complete, admissible and preferred semantics. Moreover, we investigate whether the maximum degree of joint attacks affects the expressiveness of SETAFs.

Contributions. The main contributions of our work are as follows.

- In Section 3 we provide full characterizations of the extension-based signatures of SETAFs for conflict-free, naive, stable, complete, admissible and preferred semantics. By that we characterize the exact difference in expressiveness between Dung AFs and SETAFs when considering extension-based semantics.

- In Section 4 we study k-SETAF where attacks are restricted to at most k arguments attacking another argument. Our characterizations of signatures for k-SETAFs for conflict-free, naive, stable, admissible and preferred semantics show that the degree of the allowed attacks is crucial for the expressiveness. That is, k-SETAFs form a strict hierarchy of expressiveness when considering different values for k.

Some of the technical proofs are omitted in the maint part of the paper but are provided in the corresponding appendices.

2 Preliminaries

We first introduce formal definitions of argumentation frameworks following [2, 5] and then recall the relevant work on signatures.

2.1 Argumentation Frameworks with collective attacks

Throughout the paper, we assume a countably infinite domain $\mathfrak{A}$ of possible arguments.

Definition 1. A SETAF is a pair $F = (A, R)$ where $A \subseteq \mathfrak{A}$ is finite, and $R \subseteq (2^A \setminus \emptyset) \times A$ is the attack relation. A k-SETAF is a SETAF where for all $(S, a) \in R$ we have $|S| \leq k$. The collection of all SETAFs (k-SETAFs) over $\mathfrak{A}$ is given as $AF_{\mathfrak{A}}$ ($AF^k_{\mathfrak{A}}$).

We will call 1-SETAF, i.e. SETAFs that only allow for binary attacks, Dung argumentation frameworks (AFs) as they are equivalent to the AFs introduced in [2]. We write $S \mapsto_R b$ if there is a set $S' \subseteq S$ with $(S', b) \in R$. Moreover, we write $S' \mapsto_R S$ if $S' \mapsto_R b$ for some $b \in S$. We drop subscript R in $\mapsto_R$ if there is no ambiguity.

Definition 2. Given a SETAF $F = (A, R)$, an argument $a \in A$ is *defended* (in F) by a set $S \subseteq A$ if for each $B \subseteq A$, such that $B \mapsto_R a$, also $S \mapsto_R B$. A set T of arguments is defended (in F) by S if each $a \in T$ is defended by S (in F).

Next, we introduce the semantics we study in this work. These are the naive, stable, preferred, complete, and grounded semantics, which we will abbreviate by *naive*, *stb*, *pref*, *com*, and *grd*, respectively. For a given semantics σ, $\sigma(F)$ denotes the set of extensions of F under σ.

Definition 3. Given a SETAF $F = (A, R)$, a set $S \subseteq A$ is *conflict-free* (in F), if $S' \cup \{a\} \not\subseteq S$ for each $(S', a) \in R$. We denote the set of all conflict-free sets in F as $cf(F)$. $S \in cf(F)$ is called *admissible* (in F) if S defends itself. We denote the set of admissible sets in F as $adm(F)$. For a conflict-free set $S \in cf(F)$, we say that

- $S \in naive(F)$, if there is no $T \in cf(F)$ with $T \supset S$,
- $S \in stb(F)$, if $S \mapsto a$ for all $a \in A \setminus S$,
- $S \in pref(F)$, if $S \in adm(F)$ and there is no $T \in adm(F)$ s.t. $T \supset S$,
- $S \in com(F)$, if $S \in adm(F)$ and $a \in S$ for all $a \in A$ defended by S,
- $S \in grd(F)$, if $S = \bigcap_{T \in com(F)} T$.

As shown in [5], most of the fundamental properties of Dung AFs extend to SETAFs. We have the same relations between the semantics, i.e. $stb(F) \subseteq pref(F) \subseteq com(F) \subseteq adm(F) \subseteq cf(F)$ and the grounded extension is the unique minimal complete extension for any SETAF F. Moreover, Dung's fundamental lemma generalizes to SETAF.

Lemma 1 ([5]). *Given a SETAF $F = (A, R)$, a set $B \subset A$, and arguments $a, b \in A$ that are defended by B. Then (a) $B \cup \{a\}$ is admissible in F and (b) $B \cup \{a\}$ defends b in F.*

The following result is in the spirit of Dung's fundamental lemma and is used later.

Lemma 2. *Given a SETAF $F = (A, R)$ and two sets $S, T \subseteq A$. If both S and T defend itself in F, then $S \cup T$ defends itself in F.*

Proof. Towards a contradiction assume that $S \cup T$ does not defend itself, i.e. there exists a set $B \subseteq A$ with $B \mapsto (S \cup T)$ such that $(S \cup T) \not\mapsto B$. Consider $B \mapsto S$. Since $(S \cup T) \not\mapsto B$ also $S \not\mapsto B$ and thus S does not defend itself in F which is a contradiction to the assumption. The case where $B \mapsto T$ behaves symmetrically. □

2.2 Signatures

The concept of signatures of argumentation semantics was introduced in [3] to characterize the expressiveness of Dung AFs and has been extended to other argumentation frameworks [8, 9]. Signatures characterize all possible sets of extensions, argumentation frameworks can provide for a given semantics.

Definition 4. The signature Σ_σ^k of a semantics σ is defined as

$$\Sigma_\sigma^k = \left\{\sigma(F) \mid F \in AF_{\mathfrak{A}}^k\right\}.$$

For unrestricted SETAFs we use $\Sigma_\sigma^\infty = \{\sigma(F) \mid F \in AF_{\mathfrak{A}}\}$.

For characterizing the signatures we make frequent use of the following concepts.

Definition 5. Given $\mathbb{S} \subseteq 2^{\mathfrak{A}}$, we use

(a) $Args_{\mathbb{S}}$ to denote $\bigcup_{S \in \mathbb{S}} S$;

(b) $dcl(\mathbb{S})$ to denote the *downward-closure* $\{S' \subseteq S \mid S \in \mathbb{S}\}$ of $\mathbb{S}$; and

(c) $PAtt_{\mathbb{S}}$ to denote the set of potential conflicts $\{S \subseteq Args_{\mathbb{S}} \mid S \notin dcl(\mathbb{S})\}$ in $\mathbb{S}$.

We call $\mathbb{S} \subseteq 2^{\mathfrak{A}}$ an *extension-set* (over $\mathfrak{A}$) if $Args_{\mathbb{S}}$ is finite. The *completion-sets* $\mathbb{C}_{\mathbb{S}}(E)$ of $E \subseteq Args_{\mathbb{S}}$ are given by $\mathbb{C}_{\mathbb{S}}(E) = \{S \in \mathbb{S} \mid E \subseteq S, \nexists S' \in \mathbb{S}, E \subseteq S' \subset S\}$.

As only extension-sets can appear in the signature of a semantics we will tacitly assume that all sets $\mathbb{S}$ in our characterizations are extension-sets.

Definition 6. Let $\mathbb{S} \subseteq 2^{\mathfrak{A}}$. We call $\mathbb{S}$

- *downward-closed* if $\mathbb{S} = dcl(\mathbb{S})$;
- *incomparable* if all elements $S \in \mathbb{S}$ are pairwise incomparable, i.e. for each $S, S' \in \mathbb{S}$, $S \subseteq S'$ implies $S = S'$;
- *tight* if for all $S \in \mathbb{S}$ and $a \in Args_{\mathbb{S}}$ it holds that if $S \cup \{a\} \notin \mathbb{S}$ then there exists an $s \in S$ such that $\{a, s\} \in PAtt_{\mathbb{S}}$;
- *conflict-sensitive* if for each $A, B \in \mathbb{S}$ such that $A \cup B \notin \mathbb{S}$ it holds that $\exists a, b \in A \cup B$: $\{a, b\} \in PAtt_{\mathbb{S}}$;
- *com-closed* if for each $\mathbb{T} \subseteq \mathbb{S}$: if $\{a, b\} \notin PAtt_{\mathbb{S}}$ for each $a, b \in Args_{\mathbb{T}}$, then $Args_{\mathbb{T}}$ has a unique completion-set in S , i.e. $|\mathbb{C}_{\mathbb{S}}(Args_{\mathbb{T}})| = 1$.

The main results for Dung AFs are summarized in the following theorem.

Theorem 1 ([3]). *Characterizations of the signatures for Dung AFs are as follows:*

$$\begin{aligned}
\Sigma^1_{cf} &= \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \textit{ is downward-closed and tight}\} \\
\Sigma^1_{naive} &= \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \textit{ is incomparable and } dcl(\mathbb{S}) \textit{ is tight}\} \\
\Sigma^1_{stb} &= \{\mathbb{S} \mid \mathbb{S} \textit{ is incomparable and tight}\} \\
\Sigma^1_{adm} &= \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \textit{ is conflict-sensitive and contains } \emptyset\} \\
\Sigma^1_{pref} &= \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \textit{ is incomparable and conflict-sensitive}\} \\
\Sigma^1_{com} &\subseteq \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \textit{ is com-closed and } \bigcap \mathbb{S} \in \mathbb{S}\}
\end{aligned}$$

3 Signatures of SETAFs with unrestricted collective attacks

In this section we give full characterizations of the SETAF signatures for the semantics under consideration. We start with the signatures of stable and preferred semantics. For both semantics we have that an extension cannot be a subset of another extension and thus the extension-sets of these semantics are incomparable. With the following construction we show that, in turn, each incomparable extension-set $\mathbb{S}$ can be realized under stable and preferred semantics.

Definition 7. Given an incomparable extension-set $\mathbb{S}$ containing at least one non-empty set we define the SETAF $F^{stb}_{\mathbb{S}} = (Args_{\mathbb{S}}, R^{stb}_{\mathbb{S}})$ with $R^{stb}_{\mathbb{S}} = \{(S, a) \mid S \in \mathbb{S}, a \in Args_{\mathbb{S}} \setminus S\}$.

Theorem 2. *We have* $\Sigma^\infty_{stb} = \{\mathbb{S} \mid \mathbb{S} \textit{ is incomparable}\}$ *and* $\Sigma^\infty_{pref} = \Sigma^\infty_{stb} \setminus \{\emptyset\}$.

Proof Sketch (for stable). First, as $stb(F) \subseteq pref(F)$ and the latter is incomparable by definition we have that also $stb(F)$ is incomparable for any SETAF F.

For $\mathbb{S} = \emptyset$ we can just consider the SETAF $F_\emptyset = (\{a\}, \{(\{a\}, a)\})$ with $stb(F_\emptyset) = \emptyset$. For $\mathbb{S} = \{\emptyset\}$ we can just consider the empty SETAF $F_{\{\emptyset\}} = (\{\}, \{\})$ with $stb(F_{\{\emptyset\}}) = \{\emptyset\}$. Given an incomparable set $\mathbb{S}$ containing at least one non-empty set we, show that $stb(F^{stb}_{\mathbb{S}}) = \mathbb{S}$. $stb(F^{stb}_{\mathbb{S}}) \supseteq \mathbb{S}$: Consider $S \in \mathbb{S}$. For each $a \in Args_{\mathbb{S}} \setminus S$ we have $S \mapsto a$ by construction. Moreover, as $\mathbb{S}$ is incomparable the set S is conflict-free and thus $S \in stb(F^{stb}_{\mathbb{S}})$. $stb(F^{stb}_{\mathbb{S}}) \subseteq \mathbb{S}$: Consider $S \subseteq Args_{\mathbb{S}}, S \notin \mathbb{S}$. First, if there is an $E \in \mathbb{S}$ such that $E \subset S$ then for each argument $a \in S \setminus E$ we have $E \mapsto a$ in $F^{stb}_{\mathbb{S}}$ and thus S attacks itself. Hence, such an S is not stable. Alternatively, if there is no $E \in \mathbb{S}$ such that $E \subseteq S$ then (a) S does not attack any argument and (b) there is an argument $a \in E$ that is not contained in S. Hence, S is not stable in $F^{stb}_{\mathbb{S}}$. □

By the above characterizations we can see that SETAFs are strictly more expressible than AFs for preferred and stable semantics. While for AFs we require the extension-set $\mathbb{S}$ to be tight in order to be realizable under *stb* and conflict-sensitive to be realizable under *pref*, we can realize any extension-set $\mathbb{S}$ that is just incomparable with SETAFs. We borrow an example from [6, 8] to illustrate this difference in expressiveness.

Example 1. Consider the extension-set $\mathbb{S} = \{\{a, b\}, \{b, c\}, \{a, c\}\}$. As $\mathbb{S}$ is neither tight nor conflict-sensitive there is no AF F with $stb(F) = \mathbb{S}$ or $pref(F) = \mathbb{S}$ [3]. Now consider the SETAF $G = (\{a, b, c\}, ((\{a, b\}, c), (\{a, c\}, b), (\{b, c\}, a))$. It is easy to verify that $stb(G) = pref(G) = \mathbb{S}$. ♢

Remark 1. Interestingly Σ^{∞}_{stb} coincides with the stable signature for bipolar abstract dialectical frameworks (BADF) [8, Thm. 22]. That is, although BADFs allow for strictly more notions of attacks and even allows for support it does not provide more expressiveness than SETAFs when using stable semantics. It is worth to mention that when realizing an extension-set with the construction of [8, Thm. 22] one obtains a BADF whose acceptance conditions are all anti-monotonic, i.e., when the condition holds for a model $S \subseteq A$ then it holds for each model $S' \subset S$ as well, and one can show that such an BADF can always be transformed into an equivalent SETAF.

We next consider conflict-free and naive semantics. The characteristics of conflict-free sets is that each subset is again conflict-free. We will show that this property of being downward-closed is also sufficient to realize an extension-set with a SETAF.

Definition 8. Given a non-empty extension-set $\mathbb{S}$ we define the SETAF $F^{cf}_{\mathbb{S}} = (Args_{\mathbb{S}}, R^{cf}_{\mathbb{S}})$ with $R^{cf}_{\mathbb{S}} = \{(S, a) \mid S \in \mathbb{S},\ a \in Args_{\mathbb{S}},\ S \cup \{a\} \in PAtt_{\mathbb{S}}\}$.

Lemma 3. *For each extension-set $\mathbb{S}$ we have $cf(F^{cf}_{\mathbb{S}}) = dcl(\mathbb{S})$.*

With the above result we obtain characterizations for the signatures of *cf* and *naive*.

Theorem 3. *We have*

- $\Sigma^{\infty}_{cf} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S}$ *is downward-closed*$\}$ *and*
- $\Sigma^{\infty}_{naive} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S}$ *is incomparable*$\}$.

In contrast, for realization with AFs and *cf* we require $\mathbb{S}$ to be tight and downward-closed and for *naive* we require that $\mathbb{S}$ is incomparable and that $dcl(\mathbb{S})$ is tight.

Example 2. Consider the extension-set $\mathbb{S} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}\}$. As $\mathbb{S}$ is not tight there is no AF F with $cf(F) = \mathbb{S}$. Now consider the SETAF $G = (\{a,b,c\}, ((\{a,b\},c), (\{a,c\},b), (\{b,c\},a))$. It is easy to verify that $cf(G) = \mathbb{S}$. $\Diamond$

In order the characterize the signature of admissible semantics in SETAFs we first generalize the notion of an extension-set being conflict-sensitive to SETAFs. That is, instead of requiring that if two sets A, B in the extension-set $\mathbb{S}$ whose union $A \cup B$ does not appear in $\mathbb{S}$ allow for a binary conflict, we now only require that they allow for conflicts (A, b), (B, a) with $a \in A, b \in B$.

Definition 9. A set $\mathbb{S} \subseteq 2^{\mathfrak{A}}$ is called *set-conflict-sensitive* if for each $A, B \in \mathbb{S}$ such that $A \cup B \notin \mathbb{S}$ it holds that $\exists b \in B : A \cup \{b\} \in \mathit{PAtt}_{\mathbb{S}}$. Furthermore, $\mathbb{S}$ is said to be *union-closed* if $\emptyset \in \mathbb{S}$ and each pair $A, B \in \mathbb{S}$ satisfies $A \cup B \in \mathbb{S}$. Let us also denote by $\mathit{ucl}(\mathbb{S})$ the $\subseteq$-minimal union-closed extension-set such that $\mathbb{S} \subseteq \mathit{ucl}(\mathbb{S})$.

By Lemma 2, we have that all extension-sets realizable with the admissible semantics are set-conflict-sensitive.

Lemma 4. *For any SETAF F, $\mathit{adm}(F)$ is set-conflict-sensitive and contains $\emptyset$.*

Furthermore, it turns out that $\mathbb{S}$ being set-conflict-sensitive (and containing the empty set) is also sufficient for being realizable in SETAFs under admissible semantics. The following two propositions give us some hint how to prove this claim: we reuse the conflict-free framework of Definition 8 and combine it with a framework that realizes the union-closure of the extension-set.

Proposition 1. *Let $\mathbb{S}$ be a set-conflict-sensitive extension-set that contains $\emptyset$. Then, we have that $\mathbb{S} = \mathit{dcl}(\mathbb{S}) \cap \mathit{ucl}(\mathbb{S})$.*

Proposition 2. *Let $F_1 = (A_1, R_1)$ and $F_2 = (A_2, R_2)$ be two argumentation frameworks and let $S \subseteq (A_1 \cap A_2)$ be a set of arguments. Then, (1) S is conflict-free w.r.t. $F_1 \cup F_2 = (A_1 \cup A_2, R_1 \cup R_2)$ iff S is conflict-free w.r.t. both F_1 and F_2; and (2) if S is admissible w.r.t. both F_1 and F_2, then S is admissible w.r.t. $F_1 \cup F_2 = (A_1 \cup A_2, R_1 \cup R_2)$.*

The next two lemmas analyze the SETAF F^{cf} w.r.t. admissible semantics.

Lemma 5. *Let $\mathbb{S}$ be a set-conflict-sensitive extension-set that contains $\emptyset$ and $S \subseteq \mathit{Args}_{\mathbb{S}}$ be some set of arguments such that $S = \bigcup \mathbb{T}$ for some subset $\mathbb{T} \subseteq \mathbb{S}$. Then, we have that $S \in cf(F^{cf}_{\mathbb{S}})$ implies $S \in \mathbb{S}$.*

Lemma 6. *Let $\mathbb{S}$ be a set-conflict-sensitive extension-set that contains $\emptyset$. Then, we have that $\mathbb{S} \subseteq \mathit{dcl}(\mathbb{S}) \subseteq \mathit{adm}(F^{cf}_{\mathbb{S}})$.*

Finally, we expand $F^{cf}_{\mathbb{S}}$ by additional arguments and attacks that ensure that only sets $S \in \mathbb{S}$ are admissible in the resulting SETAF $F^{adm}_{\mathbb{S}}$. In particular, for each argument a we add an argument x_a that attacks a and itself, and is only attacked by sets $S \in \mathbb{S}$.

Definition 10. Given an extension $\mathbb{S}$ set we define $F^{ucl}_{\mathbb{S}} = (A^{ucl}_{\mathbb{S}}, R^{ucl}_{\mathbb{S}})$ with $A^{ucl}_{\mathbb{S}} = \mathit{Args}_{\mathbb{S}} \cup \{x_a \mid a \in \mathit{Args}_{\mathbb{S}}\}$ and $R^{ucl}_{\mathbb{S}} = \{(\{x_a\}, a) \mid a \in \mathit{Args}_{\mathbb{S}}\} \cup \{(\{x_a\}, x_a) \mid a \in \mathit{Args}_{\mathbb{S}}\} \cup \{(S, \{x_a\}) \mid S \in \mathbb{S}$ and $a \in S\}$. We then define $F^{adm}_{\mathbb{S}} = (A^{adm}_{\mathbb{S}}, R^{adm}_{\mathbb{S}}) = (F^{cf}_{\mathbb{S}} \cup F^{ucl}_{\mathbb{S}})$.

With the following lemma we show that $F^{ucl}_{\mathbb{S}}$ can realize $ucl(\mathbb{S})$.

Lemma 7. *For every extension-set $\mathbb{S}$ that is set-conflict-sensitive and contains $\emptyset$, we have that $ucl(\mathbb{S}) \subseteq adm(F^{ucl}_{\mathbb{S}})$.*

Next we combine the results for the SETAFs $F^{cf}_{\mathbb{S}}$, $F^{ucl}_{\mathbb{S}}$ to obtain that their union $F^{adm}_{\mathbb{S}}$ realizes admissible extension-sets $\mathbb{S}$.

Lemma 8. *For every extension-set $\mathbb{S}$ that is set-conflict-sensitive and contains $\emptyset$, we have that $adm(F^{adm}_{\mathbb{S}}) = \mathbb{S}$.*

Proof. From Proposition 1, we have that $\mathbb{S} = dcl(\mathbb{S}) \cap ucl(\mathbb{S})$. Then, from Lemmas 6 and 7, we get that $\mathbb{S} \subseteq adm(F^{cf}_{\mathbb{S}}) \cap adm(F^{ucl}_{\mathbb{S}})$. Furthermore, from Proposition 2, this implies that $\mathbb{S} \subseteq adm(F^{adm}_{\mathbb{S}})$. Let us show that $adm(F^{adm}) \subseteq \mathbb{S}$ also holds. Pick any $A \in adm(F^{adm})$. Then, for every argument $a \in A$ (there is an attack $(\{x_a\}, a) \in R^{adm}$ and, so) there must be an attack $(T_a, \{x_a\}) \in R^{adm}$ with $T_a \subseteq A$. Furthermore, by construction, we also have that $T_a \in \mathbb{S}$ and $a \in T_a$. Let $\mathbb{T} = \{T_a \subseteq A \mid a \in A\} \subseteq \mathbb{S}$ and $C = \bigcup \mathbb{T}$. Then, we have that that $C = A$ and, from Lemma 5 and the fact that $A \in adm(F^{adm}_{\mathbb{S}}) \subseteq cf(F^{adm}_{\mathbb{S}}) \subseteq cf(F^{cf}_{\mathbb{S}})$, it follows that, $A \in \mathbb{S}$. □

Now we can give an exact characterization of Σ^{∞}_{adm}.

Theorem 4. $\Sigma^{\infty}_{adm} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S}$ *is set-conflict-sensitive and contains* $\emptyset\}$.

AFs require that an extension-set $\mathbb{S}$ is conflict-sensitive in order to be realizable under admissible semantics. Being set-conflict-sensitive is a strictly weaker condition as illustrated in the following example.

Example 3. Consider the extension-set $\mathbb{S} = \{\emptyset, \{a, b\}, \{b, c\}, \{a, c\}\}$. As $\{a, b, c\} \notin \mathbb{S}$ but $\{a, b\}$, $\{b, c\}, \in \mathbb{S}$ and both $\{a, c\} \notin PAtt_{\mathbb{S}}$ and $\{b, c\} \notin PAtt_{\mathbb{S}}$ the set $\mathbb{S}$ is not conflict-sensitive. Thus, there is no AF F with $adm(F) = \mathbb{S}$. Now consider the SETAF $G = (\{a, b, c\}, ((\{a, b\}, c), (\{a, c\}, b), (\{b, c\}, a))$. It is easy to verify that $adm(G) = \mathbb{S}$. ◊

Note also that the converse of Proposition 1 does not hold and that satisfying $\mathbb{S} = dcl(\mathbb{S}) \cap ucl(\mathbb{S})$ is a necessary, but not a sufficient condition. The following example illustrates this fact.

Example 4. Consider the extension-set $\mathbb{S} = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, \{a, c\}\}$. Then, we have that $dcl(\mathbb{S}) = \mathbb{S} \cup \{\{b\}, \{c\}\}$ and $ucl(\mathbb{S}) = \mathbb{S} \cup \{\{a, b, c\}\}$. It is easy to see that $\mathbb{S} = dcl(\mathbb{S}) \cap ucl(\mathbb{S})$, but that $\mathbb{S}$ is not set-conflict-sensitive: pick $A = \{a\}$ and $B = \{b, c\}$. Hence, $\mathbb{S}$ does not belong to the signature of the admissible semantics. ◊

Finally, we consider the signature of complete semantics. First, recall that the completion-sets $\mathbb{C}_{\mathbb{S}}(E)$ of a set $E \subseteq Args_{\mathbb{S}}$ are the $\subseteq$-minimal sets $S \in \mathbb{S}$ with $E \subseteq S$. Next we introduce the notion of an extension-set to be *set-com-closed* which generalizes the concept of being com-closed and allows for an exact characterization of the signature of complete semantics. The intuition is that if we pick some elements from $\mathbb{S}$ then either the union of these sets has a unique completion or we can draw an attack within this set.

Definition 11. A set $\mathbb{S} \subseteq 2^A$ is called set-com-closed iff for each $\mathbb{T}, \mathbb{U} \subseteq \mathbb{S}$ with $T = \bigcup \mathbb{T}$, $U = \bigcup \mathbb{U}$ the following holds: If $T, U \in \mathit{dcl}(\mathbb{S})$ and $|\mathbb{C}_{\mathbb{S}}(T \cup U)| \neq 1$ then there is an argument $u \in U$ such that $T \cup \{u\} \in \mathit{PAtt}_{\mathbb{S}}$.

Intuitively the set of complete extensions is set-com-closed because whenever the union of some complete extension has no conflict, by Lemma 2, then this union is admissible and there is a unique minimal complete extensions containing this admissible set. Moreover, the grounded extensions is the intersection of all complete extensions and complete as well.

Lemma 9. *For every SETAF F we have that (a) the extension-set $\mathit{com}(F)$ is set-comp-closed and (b) $\bigcap \mathit{com}(F) \in \mathit{com}(F)$.*

Our realization for complete semantics is based on the construction for the admissible semantics given in Definition 10. First, given an extension-set $\mathbb{S}$, by $\mathit{reduced}(\mathbb{S}) = \{S \setminus \bigcap \mathbb{S} \mid S \in \mathbb{S}\}$, we denote a reduced extension-set whose corresponding ground extension is empty. Let $\mathbb{S}' = \mathit{reduced}(\mathbb{S})$. We then realize $\mathbb{S}^* = \mathit{dcl}(\mathbb{S}') \cap \mathit{ucl}(\mathbb{S}') = \{ \bigcup \mathbb{T} \mid \mathbb{T} \subseteq \mathbb{S}, \bigcup \mathbb{T} \in \mathit{dcl}(\mathbb{S}') \}$ and add further attacks such that each set $E \in \mathbb{S}^*$ defends all arguments of the unique set in $\mathbb{C}_{\mathbb{S}}(E)$. In the following we use $\mathsf{C}_{\mathbb{S}}(E)$ to denote the unique element of $\mathbb{C}_{\mathbb{S}}(E)$ iff $|\mathbb{C}_{\mathbb{S}}(E)| = 1$ and the empty set otherwise.

Definition 12. Given an extension-set $\mathbb{S}$, let $\mathbb{S}' = \mathit{reduced}(\mathbb{S})$ and $\mathbb{S}^* = \mathit{dcl}(\mathbb{S}') \cap \mathit{ucl}(\mathbb{S}')$. Then, by $F^{com}_{\mathbb{S}} = (A^{adm}_{\mathbb{S}}, R^{com}_{\mathbb{S}})$ we denote a SETAF with $R^{com}_{\mathbb{S}} = R^{adm}_{\mathbb{S}^*} \cup R'$ and where $R' = \{(A \cup B, x_a) \mid A, B \in \mathbb{S}' \setminus \{\emptyset\}, a \in \mathsf{C}_{\mathbb{S}'}(A \cup B)\}$.

One can show that this construction realizes extension-sets with complete semantics whenever possible.

Lemma 10. *For every extension-set $\mathbb{S}$ that is set-comp-closed and satisfies $\bigcap \mathbb{S} \in \mathbb{S}$, we have that $\mathit{com}(F^{com}_{\mathbb{S}}) = \mathbb{S}$.*

This now gives a complete characterization of the signature for complete semantics.

Theorem 5. $\Sigma^{\infty}_{com} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S}$ *is set-comp-closed and* $\bigcap \mathbb{S} \in \mathbb{S}\}$.

Notice that when considering AFs not all extension-sets that are com-closed and satisfy $\bigcap \mathbb{S} \in \mathbb{S}$ are realizable with the complete semantics and a full characterization of complete semantics is an open problem [3]. This is in contrast to the above result which provides a full characterizations for SETAFs.

Example 5. Consider the extension-set $\mathbb{S} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b, c\}, \{a, d, e\}, \{b, d, f\}, \{x, c\}, \{x, d\}\}$ which cannot be realized with AFs [3, Example 8]. It is easy to verify that $\mathbb{S}$ set-comp-closed and thus $\mathit{com}(F^{com}_{\mathbb{S}}) = \mathbb{S}$. $\Diamond$

4 Signatures of SETAFs with Bounded Degree Collective Attacks

We now investigate how the degree of collective attacks affects the expressiveness, i.e. we study k-SETAFs. Notice that in all the constructions of the last section we used attacks of unbounded degree, i.e. the actual degree typically depended on the size of the extensions.

We first generalize the properties used in our signatures by adding a parameter k.

Definition 13. The possible conflicts in a k-SETAF w.r.t. an extension-set $\mathbb{S}$ are defined as $\mathit{PAtt}_{\mathbb{S}}^{k} = \{S \subseteq \mathit{Args}_{\mathbb{S}} \mid |S| \leq k+1 \text{ and } S \notin \mathit{dcl}(\mathbb{S})\}$. An extension-set $\mathbb{S} \subseteq 2^{\mathfrak{A}}$ is *k-tight* if for all $S \in \mathbb{S}$ and $a \in \mathit{Args}_{\mathbb{S}}$ it holds that if $S \cup \{a\} \notin \mathbb{S}$ then there exists a set $S' \subseteq S$, such that $S' \cup \{a\} \in \mathit{PAtt}_{\mathbb{S}}^{k}$.

For $k = 1$ the notion of k-tight corresponds to the notion of tight on Dung AFs (see Definition 6) while for $k \geq \mathit{Args}_{\mathbb{S}}$ the notion of k-tight simplifies to: for all $S \in \mathbb{S}$ and $a \in \mathit{Args}_{\mathbb{S}}$ either $S \cup \{a\} \in \mathbb{S}$ or there is no $S' \in \mathbb{S}$ with $S \cup \{a\} \subseteq S'$. Thus, $\mathbb{S}$ being ∞-tight is implied by both $\mathbb{S}$ being incomparable or $\mathbb{S}$ being downward-closed.

We start with presenting our results for the signatures for conflict-free and naive semantics. We already know that conflict-free extension-sets must be downward-closed. In k-SETAFs we additionally have that they must be k-tight which reflects that if $S \cup \{a\}$ is not conflict-free there must be an attack in the set of degree at most k. The following construction allows us to also realize such extension-sets.

Definition 14. For downward-closed and k-tight extension-sets $\mathbb{S}$, let $F_{\mathbb{S}}^{cf,k} = (\mathit{Args}_{\mathbb{S}}, R_{\mathbb{S}}^{cf,k})$ be the k-SETAF with $R_{\mathbb{S}}^{cf,k} = \{(S, a) \mid S \subseteq \mathit{Args}_{\mathbb{S}}, a \in \mathit{Args}_{\mathbb{S}}, S \cup \{a\} \in \mathit{PAtt}_{\mathbb{S}}^{k}\}$.

One can show that (a) for each $\mathbb{S}$ that is downward-closed and k-tight we have that $\mathit{cf}(F_{\mathbb{S}}^{cf,k}) = \mathbb{S}$ and (b) for each $\mathbb{S}$ that is incomparable and whose downward-closure is k-tight we have that $\mathit{naive}(F_{\mathbb{S}}^{cf,k}) = \mathbb{S}$.

Theorem 6. *We have that*

- $\Sigma_{cf}^{k} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S}$ *is downward-closed and k-tight*$\}$ *and*
- $\Sigma_{naive}^{k} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S}$ *is incomparable and* $\mathit{dcl}(\mathbb{S})$ *is k-tight*$\}$.

The following example shows that the expressiveness of conflict-free and naive semantics strictly increases with the degree k of the attacks.

Example 6. Consider the argument set $A = \{a_1, a_2, \ldots, a_{k+1}, a_{k+2}\}$ and the extension-sets $\mathbb{S} = \{S \subseteq A \mid |S| \leq k+1\}$ and $\mathbb{T} = \{S \subseteq A \mid |S| = k+1\}$. We have that $\mathbb{S}$ is not k-tight, as $A \notin \mathbb{S}$, but for $S = \{a_1, a_2, \ldots, a_{k+1}\}$ we have that every $S' \subset \{a_1, a_2, \ldots, a_{k+1}\}$ satisfies $S' \cup \{a_{k+2}\} \in \mathbb{S}$ and thus $S' \cup \{a_{k+2}\} \notin \mathit{PAtt}_{\mathbb{S}}^{k}$. Note that $S \cup \{a_{k+2}\} \notin \mathit{PAtt}_{\mathbb{S}}^{k}$ because $|S \cup \{a_{l+2}\}| > k+1$. Hence, $\mathbb{S}$ cannot be realized as conflict-free sets of any k-SETAF. However, one can easily verify that $\mathbb{S}$ is $(k+1)$-tight and thus can be realized as conflict-free sets of some $(k+1)$-SETAF. Moreover, as $\mathit{dcl}(\mathbb{T}) = \mathbb{S}$ we have that $\mathit{dcl}(\mathbb{T})$ is not k-tight, i.e. $\mathbb{T}$ cannot be realized as naive sets of a k-SETAF, and $\mathit{dcl}(\mathbb{T})$ is $(k+1)$-tight, i.e. $\mathbb{T}$ can be realized as naive sets of a $(k+1)$-SETAF. $\Diamond$

Next we consider the stable signature for k-SETAFs. Again, the set of stable extensions of a k-SETAF must be k-tight reflecting the fact that each argument which is not in an extension S must be attacked by S via a degree k attack. The following construction expands $F_{\mathbb{S}}^{cf,k}$ by arguments x_s that eliminate unwanted naive extensions of $F_{\mathbb{S}}^{cf,k}$.

Definition 15. When given an extension-set $\mathbb{S}$ that is incomparable and k-tight we can construct the k-SETAF $F_k^{stb} = (A, R)$ based on $F_{\mathbb{S}}^{cf,k}$ as follows:

$$A = \mathit{Args}_{\mathbb{S}} \cup \{x_S \mid S \notin \mathbb{S} \text{ and } S \in \mathit{naive}(F_{\mathbb{S}}^{cf,k})\}$$
$$R = R_{\mathbb{S}}^{cf,k} \cup \{(\{a\}, x_S), (\{x_S\}, x_S) \mid a \in \mathit{Args}_{\mathbb{S}} \setminus S\}$$

One can show that for each $\mathbb{S}$ that is incomparable and k-tight we have that $\mathit{stb}(F_{\mathbb{S}}^{stb,k}) = \mathbb{S}$ by building on Theorem 6 and using similar arguments as in [3, Prop. 7].

Theorem 7. $\Sigma_{stb}^k = \{\mathbb{S} \mid \mathbb{S}$ *is incomparable and k-tight*$\}$.

The above theorem gives a strict hierarchy of signatures Σ_{stb}^k which is illustrated in the following example.

Example 7. Consider the argument set $A = \{a_1, a_2, \ldots, a_{k+1}, a_{k+2}\}$ and the extension-set $\mathbb{T} = \{S \subseteq A \mid \ |S| = k+1\}$ as in Example 6. Recall that $\mathbb{T}$ was not realizable by the naive semantics because $\mathit{dcl}(\mathbb{T})$ was not k-tight. It results that $\mathbb{T}$ is itself not k-tight either. Note that $A \notin \mathbb{T}$, but for $\{a_1, a_2, \ldots, a_{k+1}\} \in \mathbb{S}$ we have that any $S \subset \{a_1, a_2, \ldots, a_{k+1}\}$ satisfies $S \cup \{a_{k+2}\} \in \mathit{dcl}(\mathbb{T})$ and thus $S \cup \{a_{k+2}\} \notin \mathit{PAtt}_{\mathbb{T}}^k$. Hence, $\mathbb{T}$ cannot be realized as stable extensions of a k-SETAF. However, one can easily verify that $\mathbb{T}$ is $(k+1)$-tight and thus can be realized as stable extensions of a $(k+1)$-SETAF. $\diamondsuit$

Note that, for incomparable $\mathbb{S}$, whenever $\mathit{dcl}(\mathbb{S})$ is k-tight, also $\mathbb{S}$ is k-tight. Hence, for k-SETAFs, the stable semantics is more expressible than the naive semantics. We next show that stable semantics is indeed strictly more expressive than naive semantics.

Example 8. Consider the sets of arguments $X = \{x_1, \ldots x_{k+1}\}$, $Y = \{y_1, \ldots y_{k+1}\}$ additional arguments a, b and the extension-set $\mathbb{S} = \{X \cup \{a\}\} \cup \{\{b, y_j\} \cup X \setminus \{x_j\} \mid 1 \leq j \leq k+1\}$. The set $\mathbb{S}$ is k-tight as $\{a, b\}, \{a, y_i\}, \{y_i, y_j\}, \{x_i, y_i\} \in \mathit{PAtt}_{\mathbb{S}}^k$. On the other hand, $\mathit{dcl}(\mathbb{S})$ is not k-tight as for the set $X \in \mathit{dcl}(\mathbb{S})$ there is no $X' \subseteq X$ such that $|X'| \leq k$ and $X' \cup \{b\} \in \mathit{PAtt}_{\mathbb{S}}^k$. That is, the extension-set $\mathbb{S}$ can be realized with a k-SETAF under stable semantics but not with a k-SETAF under the naive semantics. $\diamondsuit$

Finally, we consider the signatures of the admissible and preferred semantics for k-SETAFs. It turns out that a simple generalization of set-conflict-sensitive is not sufficient to characterize admissible extension-sets. We thus introduce the more involved notion of *k-defensive*, which simplifies to set-conflict-sensitive for $k = \infty$ and to conflict-sensitive for $k = 1$.

Definition 16. A set $\mathbb{S} \subseteq 2^{\mathfrak{A}}$ is called *k-defensive* if there exists a set P of pairs (A_S^i, b) with $A_S^i \subseteq S \in \mathbb{S}$ and $b \in \mathit{Args}_{\mathbb{S}} \setminus S$ and $A_S^i \cup \{b\} \in \mathit{PAtt}_{\mathbb{S}}^k$, such that (i) for $S, S' \in \mathbb{S}$ with $S \cup S' \notin \mathbb{S}$ there is a pair $(A_S^i, b) \in P$ with $b \in S'$, and (ii) for each $(A_S^i, b) \in P$ with $b \in S'$ there is $(A_{S'}^j, a) \in P$ with $a \in A_S^i$.

Whenever the union of two admissible sets is not admissible then there (i) must be an attack of degree $\leq k$ in this union and (ii) each admissible set must defend itself against all attacks we introduce to establish (i), again using only attacks of degree $\leq k$.

Lemma 11. *For any SETAF F we have that $adm(F)$ is k-defensive and contains $\emptyset$.*

Remark 2. For $k = 1$, we can make all the elements of P symmetric and thus the second condition of the above definition holds trivially true. That is, the notion of 1-defensive reduces to being conflict-sensitive, cf. Definition 6. For unbounded k, each set (A^i_S, b) can be replaced by (S, b) without violating either of the two conditions in the above definition. Condition (i) then simplifies to for $S, S' \in \mathbb{S}$ with $S \cup S' \notin \mathbb{S}$ there is a $b \in S'$ with (S, b). Then condition (ii) is trivially satisfied and set-defensive reduces to being set-conflict-sensitive.

Similarly as done in Section 3 for SETAFs of unbounded attack degree, we build the k-SETAF for the admissible semantics with several modules, starting with the module that exploits conflict-freeness.

Definition 17. When given a k-defensive extension-set $\mathbb{S}$ and a set P that meets the conditions of Definition 16 we define the k-SETAF $F^{cf,k}_{\mathbb{S},P} = (Args_{\mathbb{S}}, P)$.

We are now able to obtain similar results for this module as for the corresponding module in general SETAFs.

Lemma 12. *Let $\mathbb{S}$ be a k-defensive signature that contains $\emptyset$, P be some set that meets the conditions of Definition 16 and $S \subseteq Args_{\mathbb{S}}$ be some set of arguments such that $S = \bigcup \mathbb{T}$ for some subset $\mathbb{T} \subseteq \mathbb{S}$. Then, we have that $S \in cf(F^{cf,k}_{\mathbb{S},P})$ implies $S \in \mathbb{S}$.*

Lemma 13. *Let $\mathbb{S}$ be a k-defensive signature with $\emptyset \in \mathbb{S}$. Then, $\mathbb{S} \subseteq dcl(\mathbb{S}) \subseteq adm(F^{cf,k}_{\mathbb{S},P})$.*

Towards our defense module we recall the notion of defense-formulas from [3].

Definition 18 ([3])**.** Given an extension-set $\mathbb{S}$, the *defense-formula* $\mathcal{D}^{\mathbb{S}}_a$ of an argument $a \in Args_{\mathbb{S}}$ in $\mathbb{S}$ is defined as $\bigvee_{S\in\mathbb{S} \text{ s.t. } a\in S} \bigwedge_{s\in S\setminus\{a\}} s$.
$\mathcal{D}^{\mathbb{S}}_a$ given as (a logically equivalent) CNF is called *CNF-defense-formula* $\mathcal{CD}^{\mathbb{S}}_a$ of a in $\mathbb{S}$.

The defense formula $\mathcal{D}^{\mathbb{S}}_a$ tells us which arguments must be in the extension in order to defend the argument a. We can exploit this by using the following technical lemma.

Lemma 14 ([3])**.** *Given an extension-set $\mathbb{S}$ and an argument $a \in Args_{\mathbb{S}}$, then for each $S \subseteq Args_{\mathbb{S}}$ with $a \in S$: $(S \setminus \{a\})$ is a model of $\mathcal{D}^{\mathbb{S}}_a$ (resp. $\mathcal{CD}^{\mathbb{S}}_a$) iff there exists an $S' \subseteq S$ with $a \in S'$ such that $S' \in \mathbb{S}$.*

For our defense module we adjust the corresponding parts from the canonical defense-argumentation-framework in [3] to our setting with k-SETAFs.

Definition 19. Given an extension-set $\mathbb{S}$, we call $F_{\mathbb{S}}^{def} = (A_{\mathbb{S}}^{def}, R_{\mathbb{S}}^{def})$ with

$$A_{\mathbb{S}}^{def} = Args_{\mathbb{S}} \cup \bigcup_{a \in Args_{\mathbb{S}}} \{\alpha_{a\gamma} \mid \gamma \in \mathcal{CD}_a^{\mathbb{S}}\}$$
$$R_{\mathbb{S}}^{def} = \bigcup_{a \in Args_{\mathbb{S}}} \{(\{b\}, \alpha_{a\gamma}), (\{\alpha_{a\gamma}\}, \alpha_{a\gamma}), (\{\alpha_{a\gamma}\}, a) \mid \gamma \in \mathcal{CD}_a^{\mathbb{S}}, b \in \gamma\}$$

the *defense-argumentation-framework* of $\mathbb{S}$, and let $F_{\mathbb{S},P}^{adm,k} = F_{\mathbb{S},P}^{cf,k} \cup F_{\mathbb{S}}^{def}$.

We next show that this defense framework ensures that only sets in $\mathbb{S}$ or the union of such sets are admissible.

Lemma 15. *For every extension-set $\mathbb{S}$ that contains $\emptyset$, we have that $S \in adm(F_{\mathbb{S}}^{def})$ iff $S = \bigcup \mathbb{T}$ for some $\mathbb{T} \subseteq \mathbb{S}$.*

When combining the two modules to a SETAF $F_{\mathbb{S},P}^{adm,k}$ by the Lemmas 12, 13 and Lemma 15 we get a SETAF that realizes admissible extension-sets.

Lemma 16. *For every extension-set $\mathbb{S}$ that is k-defensive and contains $\emptyset$, $adm(F_{\mathbb{S},P}^{adm}) = \mathbb{S}$.*

We now can state the exact characterization of the admissible signature in k-SETAFs.

Theorem 8. *We have that*

- $\Sigma_{adm}^{k} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S}$ *is k-defensive and contains* $\emptyset\}$ *and*
- $\Sigma_{pref}^{k} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S}$ *is incomparable and k-defensive*$\}$.

Notice that we omitted complete semantics for k-SETAFs. This is due to the fact that finding an exact characterization is a hard problem (open even for Dung AFs) and our under-/over-approximations are rather tedious.

5 Discussion and Related Work

Discussion of our Results. In this work we characterized the signatures of SETAFs and SETAFs with bounded degree attacks. Our results on signatures of general SETAFs are summarized in the following theorem which is by Theorems 2-5.

Theorem 9. *Characterizations of the signatures for general SETAFs are as follows:*

$$\begin{aligned}
\Sigma_{cf}^{\infty} &= \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \textit{ is downward-closed}\} \\
\Sigma_{naive}^{\infty} &= \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \textit{ is incomparable}\} \\
\Sigma_{stb}^{\infty} &= \{\mathbb{S} \mid \mathbb{S} \textit{ is incomparable}\} \\
\Sigma_{adm}^{\infty} &= \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \textit{ is set-conflict-sensitive and contains } \emptyset\} \\
\Sigma_{pref}^{\infty} &= \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \textit{ is incomparable}\} \\
\Sigma_{com}^{\infty} &= \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \textit{ is set com-closed and } \bigcap \mathbb{S} \in \mathbb{S}\}
\end{aligned}$$

Our results on signatures of k-SETAFs are summarized in the following theorem which is by Theorems 6-8.

Theorem 10. *Characterizations of the signatures for k-SETAFs are as follows:*

$$\Sigma^k_{cf} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \text{ is downward-closed and } k\text{-tight}\}$$
$$\Sigma^k_{naive} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \text{ is incomparable and } dcl(\mathbb{S}) \text{ is } k\text{-tight}\}$$
$$\Sigma^k_{stb} = \{\mathbb{S} \mid \mathbb{S} \text{ is incomparable and } k\text{-tight}\}$$
$$\Sigma^k_{adm} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \text{ is } k\text{-defensive and contains } \emptyset\}$$
$$\Sigma^k_{pref} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \text{ is incomparable and } k\text{-defensive}\}$$

We highlight some interesting findings: (1) For all the semantics SETAFs are strictly more expressive than AFs (even for degree 2 attacks). (2) For SETAFs the signatures of stable, preferred and naive coincide which is in contrast to Dung AFs and k-SETAFs where we have strict subset relations, i.e. $\Sigma^k_{naive} \subset \Sigma^k_{stb} \setminus \{\emptyset\} \subset \Sigma^k_{pref}$ for $1 \leq k < \infty$. (3) When considering the signatures of k-SETAFs the expressiveness strictly increases with k for all of the semantics. (4) For stable semantics the signature of SETAFs coincides with the signature of Abstract Dialectical Frameworks, which allow for way more complex relations between arguments.

Related Work. The work closest to ours is by Linsbichler et al. [4] and by Polberg [6]. The former studies SETAFs as a sub-class of ADFs with 3-valued semantics. In order to meet the 3-valued setting the extension-based semantics of SETAFs are redefined as 3-valued semantics. They then provide an algorithmic framework that tests whether a given set of 3-valued extensions can be realized as SETAF. Their results allow to compare the expressiveness of admissible, complete, preferred, and stable semantics in AFs, SETAFs, and ADFs, but do not provide an explicit characterization of the sets that can be realized as SETAFs. Moreover, the setting with 3-valued semantics is more restrictive than the extension-based view and thus these results do not translate to the original definition of Dung AF and SETAF semantics. The work of Polberg [6, Section 4.4.1] studies translations between different abstract argumentation formalisms in the extension-based setting. It already shows that there are certain sets of extensions that can be realized by SETAFs but cannot be realized with AFs, in order to show that certain translations are impossible. However, the exact expressiveness of SETAFs is not investigated any further.

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A Proofs of Section 3

A.1 Signatures of stable and preferred semantics

We split the proof of Theorem 2 into two propositions.

Proposition 3. $\Sigma^{\infty}_{stb} = \{\mathbb{S} \mid \mathbb{S} \textit{ is incomparable}\}$

Proof. First, as $stb(F) \subseteq pref(F)$ and the latter is incomparable by definition we have that also $stb(F)$ is incomparable for any SETAF F.

For $\mathbb{S} = \emptyset$ we can just consider the SETAF $F_\emptyset = (\{a\}, \{(\{a\}, a)\})$ with $stb(F_\emptyset) = \emptyset$. For $\mathbb{S} = \{\emptyset\}$ we can just consider the empty SETAF $F_{\{\emptyset\}} = (\{\}, \{\})$ with $stb(F_{\{\emptyset\}}) = \{\emptyset\}$. Given an incomparable set $\mathbb{S}$ containing at least one non-empty set we construct the SETAF $F^{stb} = (A, R)$ (cf. Definition 7). We have that $stb(F^{stb}) = \mathbb{S}$. $stb(F^{stb}) \supseteq \mathbb{S}$: Consider $S \in \mathbb{S}$. For each $a \in Args_{\mathbb{S}} \setminus S$ we have $(S, a) \in R$ by construction. Moreover, as $\mathbb{S}$ is incomparable the set S is conflict-free and thus $S \in stb(F^{stb})$. $stb(F^{stb}) \subseteq \mathbb{S}$: Consider $S \subseteq Args_{\mathbb{S}}, S \notin \mathbb{S}$. First, if there is an $E \in \mathbb{S}$ such that $E \subset S$ then for each argument $a \in S \setminus E$ we have $(E, a) \in R$ and thus S attacks itself. Hence, such an S is not stable. Alternatively, if there is no $E \in \mathbb{S}$ such that $E \subseteq S$ then (a) S does not attack any argument and (b) there is an argument $a \in E$ that is not contained in S. Hence, S is not stable in F^{stb}. □

Next consider preferred semantics. By definition the set of preferred extensions in incomparable. We next show that being incomparable is also sufficient for an extension-set $\mathbb{S}$ to be realizable under preferred semantics.

Proposition 4. $\Sigma^{\infty}_{pref} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \textit{ is incomparable}\}$

Proof. First, $pref(F)$ is incomparable and non-empty by definition (for any SETAF F). For $\mathbb{S} = \{\emptyset\}$ we can just consider the empty SETAF $F_{\{\emptyset\}} = (\{a\}, \{(\{a\}), a\})$ with $pref(F_{\{\emptyset\}}) = \{\emptyset\}$. Given an incomparable set $\mathbb{S}$ containing at least one non-empty set we again consider the SETAF $F^{stb} = (A, R)$ (cf. Definition 7). We have that $pref(F^{stb}) = \mathbb{S}$.

$pref(F^{stb}) \supseteq \mathbb{S}$: Consider $S \in \mathbb{S}$. For each $a \in Args_{\mathbb{S}} \setminus S$ we have $(S, a) \in R$ by construction. Moreover, as $\mathbb{S}$ is incomparable the set S is conflict-free and thus $S \in stb(F^{stb})$.

$pref(F^{stb}) \subseteq \mathbb{S}$: Consider $S \subseteq Args_{\mathbb{S}}, S \notin \mathbb{S}$. First, if there is an $E \in \mathbb{S}$ such that $E \subset S$, then there is an argument $a \in S \setminus E$ such that $(E, a) \in R$ and thus S attacks itself and thus is neither conflict-free nor preferred. Thus let us consider the case where there is no $E \in \mathbb{S}$ such that $E \subseteq S$. Then S does not attack any argument. Notice that by construction all arguments, except those arguments contained in all sets $S \in \mathbb{S}$ (we call them skeptically accepted arguments), are attacked by at least one set $S \in \mathbb{S}$. If S contains an argument that is not skeptically accepted, S cannot be admissible as it is attacked and has no outgoing attacks. On the other hand side if S only contains skeptically accepted arguments then it is a strict subset of some set in $\mathbb{S}$ and thus cannot be $\subseteq$-maximal among the admissible sets. That is, $S \notin pref(F^{stb})$. □

A.2 Signatures of grounded, conflict-free and naive semantics

First, we consider grounded semantics. Grounded semantics, in SETAFs as well as in AFs, is a unique status semantics, i.e. it always yields a unique extensions. Thus it can only realize extension-sets that contain exactly one extension.

Proposition 5. $\Sigma_{grd}^{\infty} = \{\mathbb{S} \mid \ |\mathbb{S}| = 1\}$

Proof. The grounded semantics always proposes a unique extension. An extension-set $\mathbb{S} = \{S\}$ single set can be realized by the SETAF with arguments S and no attacks, i.e. by the SETAF $(S, \emptyset)$. □

Next consider conflict-free sets. We have that if a set is conflict-free then also all it subsets are conflict-free. As we show next, the fact that an extension-set is downward-closed is also sufficient to realize it with a SETAF.

Lemma 3. (restated) For each extension-set $\mathbb{S}$ we have $\mathit{cf}(F^{cf}) = \mathit{dcl}(\mathbb{S})$.

Proof. Consider the SETAF F^{cf} (see Definition 8) and let us show first that $\mathit{cf}(F^{cf}) \supseteq \mathbb{S}$. Pick any $S \in \mathbb{S}$ and any attack $(S', a) \in R$ with $S' \subseteq S$. By construction, we have that $(S' \cup \{a\}) \in \mathit{PAtt}_{\mathbb{S}}$ and, thus, that $(S' \cup \{a\}) \not\subseteq S$. Hence, since $S' \subseteq S$, it follows that $a \notin S$ and that S is conflict-free. Hence, we have that $\mathit{cf}(F^{cf}) \supseteq \mathbb{S}$ and that $\mathit{dcl}(\mathit{cf}(F^{cf})) \supseteq \mathit{dcl}(\mathbb{S})$. Furthermore, from Proposition 6, we also have that $\mathit{cf}(F^{cf}) = \mathit{dcl}(F^{cf})$ and, thus, we get $\mathit{cf}(F^{cf}) \supseteq \mathit{dcl}(\mathbb{S})$. Let us show now that $\mathit{cf}(F) \subseteq \mathit{dcl}(\mathbb{S})$ also holds. Pick $S \in \mathit{PAtt}_{\mathbb{S}}$, some argument $a \in S$ and let $S' = S \setminus \{a\}$. Then, by construction $(S', a) \in R$ and, thus, S is not conflict-free. □

Proposition 6. $\Sigma_{cf}^{\infty} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \text{ is downward-closed}\}$

Proof. By definition, if a set is conflict-free then all its subsets are conflict-free as well. Thus, we have that $\mathit{cf}(F)$ is downward closed for all SETAFs F. We next consider the SETAF F^{cf} (see Definition 8). By Lemma 3 we have $\mathit{cf}(F^{cf}) = \mathit{dcl}(\mathbb{S})$ and thus also $\mathit{cf}(F^{cf}) = \mathbb{S}$. □

Proposition 7. $\Sigma_{naive}^{\infty} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \text{ is incomparable}\}$

Proof. By definition, if a set is naive then all it is a maximal conflict-free set. Thus, we have that $\mathit{naive}(F)$ is incomparable for all SETAFs F. We next consider the SETAF F^{cf} (see Definition 8). and show that $\mathit{naive}(F^{cf}) = \mathbb{S}$. First notice that by Lemma 3 we have $\mathit{cf}(F) = \mathit{dcl}(\mathbb{S})$. As $\mathbb{S}$ contains exactly the $\subseteq$-maximal elements of $\mathit{dcl}(\mathbb{S})$ and the naive extension of F are the $\subseteq$-maximal elements of $\mathit{cf}(F)$ we obtain $\mathit{naive}(F) = \mathbb{S}$. □

A.3 Signature of Admissible Semantics

This section is devoted to proof of Theorem 4.

Theorem 4. (restated) $\Sigma_{adm}^{\infty} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S}$ is set-conflict-sensitive and contains $\emptyset\}$.

We first show that the conditions of our characterization are indeed necessary.

Lemma 4. (restated) For any SETAF F we have that $\mathit{adm}(F)$ is set-conflict-sensitive and contains $\{\emptyset\}$.

Proof. First, notice that the empty set is always admissible. Next assume there are two admissible sets A, B such that the set $C = A \cup B$ is not admissible. By Lemma 2 the set C defends itself against all attackers and thus there must be a conflict in C, i.e. there exists an attack $(S, a) \in R$ with $S \subseteq C$ and $a \in A$.

- If $a \in A$ then, as A is conflict-free, $S \cap B \neq \emptyset$. Moreover, as A is admissible it has to defend itself against (S, a), i.e. there is an attack (S', b) with $S' \subseteq A$ and $b \in S \cap B$. Hence, we have $S' \cup \{b\} \in \mathit{PAtt}_{\mathit{adm}(F)}$.
- If $a \in B$ then, as B is conflict-free, $S \cap A \neq \emptyset$. Moreover, as B is admissible it has to defend itself against (S, a), i.e. there is an attack (S^*, c) with $S^* \subseteq B$ and $c \in S \cap A$. Now, as A is admissible as well, there is also an attack (S', b) with $S' \subseteq A$ and $b \in S^* \subseteq B$. Hence, we have $S' \cup \{b\} \in \mathit{PAtt}_{\mathit{adm}(F)}$.

We obtain that $\mathit{adm}(F)$ is set-conflict-sensitive. □

Proposition 1 (restated) Let $\mathbb{S}$ be a set-conflict-sensitive extension-set that contains $\emptyset$. Then, we have that $\mathbb{S} = \mathit{dcl}(\mathbb{S}) \cap \mathit{ucl}(\mathbb{S})$.

Proof. Pick any set-conflict-sensitive $\mathbb{S}$ and let $\mathbb{S}' = \mathit{dcl}(\mathbb{S})$ and $\mathbb{S}''$ be the union closure of $\mathbb{S}''$. By construction, we have that $\mathbb{S}'$ is downward-closed, that $\mathbb{S}''$ is union-closed, that $\emptyset \in \mathbb{S}' \cap \mathbb{S}''$ and that $\mathbb{S}' \cap \mathbb{S}'' \supseteq \mathbb{S}$. Hence, it only remains to be shown that $\mathbb{S}' \cap \mathbb{S}'' \subseteq \mathbb{S}$ also holds. Suppose for the sake of contradiction that there is some set $S \in ((\mathbb{S}' \cap \mathbb{S}'') \setminus \mathbb{S})$. Since $S \in \mathbb{S}'$, by construction, S must be of the form $S = A \cup B$ with $A, B \in \mathbb{S}$ and, since $\mathbb{S}$ is set-conflict-sensitive, $S \notin \mathbb{S}$ implies that there is some $b \in B$ such that $(A \cup \{b\}) \in \mathit{PAtt}_{\mathbb{S}}$. Furthermore, since $S \in \mathbb{S}'$, there is also some $S' \in \mathbb{S}$ such that $S \subseteq S'$ and, thus, we have $(A \cup \{b\}) \subseteq (A \cup B) \subseteq S \subseteq S'$ which is a contradiction. Hence, it must be that $\mathbb{S}' \cap \mathbb{S}'' \subseteq \mathbb{S}$ and $\mathbb{S}' \cap \mathbb{S}'' = \mathbb{S}$ hold. □

Now assuming that an extension-set satisfies the conditions of our characterization we step-wise construct an AF realizing $\mathbb{S}$. We start with the sub-AF that exploits the conflict-free condition of admissible semantics to rule out set $A \cup B \notin \mathbb{S}$ for $A, B \in \mathbb{S}$.

The *proof of Proposition 2* is by the following two lemmas.

Lemma 17. *Let $F_1 = (A_1, R_1)$ and $F_2 = (A_2, R_2)$ be two frameworks and let $S \subseteq (A_1 \cap A_2)$ be a set of arguments. Then, S is conflict-free w.r.t. $F_1 \cup F_2 = (A_1 \cup A_2, R_1 \cup R_2)$ iff S is conflict-free w.r.t. both F_1 and F_2.*

Proof. Pick any $(A, b) \in (R_1 \cup R_2)$. Hence, $(A, b) \in R_i$ for some $i \in \{1, 2\}$ and, since S is conflict-free w.r.t. both F_1 and F_2 (resp. $F_1 \cup F_2$), it follows that $(A \cup \{b\}) \not\subseteq S$. That is, for every attack $(A, b) \in (R_1 \cup R_2)$, we have that $(A \cup \{b\}) \not\subseteq S$ and, thus, that in S is conflict-free w.r.t. $F_1 \cup F_2$ (resp. both F_1 and F_2). □

Lemma 18. *Let $F_1 = (A_1, R_1)$ and $F_2 = (A_2, R_2)$ be two frameworks and let $S \subseteq (A_1 \cap A_2)$ be a set of arguments. If S is admissible w.r.t. both F_1 and F_2, then S is admissible w.r.t. $F_1 \cup F_2 = (A_1 \cup A_2, R_1 \cup R_2)$.*

Proof. First, note that since S is admissible w.r.t. both F_1 and F_2, it is also conflict-free w.r.t. both F_1 and F_2 and, from Lemma 17, this implies that S is conflict-free w.r.t. $F_1 \cup F_2$. Let us show us that S also defends itself. Pick any $b \in S$ and $(A, b) \in (R_1 \cup R_2)$. Hence, $(A, b) \in R_i$ for some $i \in \{1, 2\}$ and, since S is admissible w.r.t. both F_1 and F_2, it follows that there is $C \subseteq S$ and $a \in A$ such that $(C, a) \in R_i$. That is, for every argument $b \in S$ and attack $(A, b) \in (R_1 \cup R_2)$, there is some $C \subseteq S$ and $a \in A$ such that $(C, a) \in (R_1 \cup R_2)$ and, thus, that in S is admissible w.r.t. $F_1 \cup F_2$. □

Lemma 5. (restated) Let $\mathbb{S}$ be a set-conflict-sensitive signature that contains $\emptyset$ and $S \subseteq \mathit{Args}_{\mathbb{S}}$ be some set of arguments such that $S = \bigcup \mathbb{T}$ for some subset $\mathbb{T} \subseteq \mathbb{S}$. Then, we have that $S \in \mathit{cf}(F^{cf}_{\mathbb{S}})$ implies $S \in \mathbb{S}$.

Proof. Let us define $\mathbb{A} \subseteq \mathbb{T}$ such that $\bigcup \mathbb{A} \in \mathbb{S}$ and there is no $\mathbb{A}' \subseteq \mathbb{T}$ such that $\mathbb{A} \subset \mathbb{A}'$ and $\bigcup \mathbb{A}' \in \mathbb{S}$. Note that such $\mathbb{A}$ always exists because $\bigcup \emptyset = \emptyset \in \mathbb{S}$. We also define $A = \bigcup \mathbb{A}$. Towards a contradiction assume that $\mathbb{A} \subset \mathbb{T}$ and pick any $B \in \mathbb{T} \setminus \mathbb{A}$. Then, by construction, we have that $A, B \in \mathbb{S}$ and that $(A \cup B) \notin \mathbb{S}$. Furthermore, since $\mathbb{S}$ is set-conflict-sensitive, it follows that there is $b \in B$ such that $(A \cup \{b\}) \in \mathit{PAtt}_{\mathbb{S}}$. This implies that there is an attack $(A, b) \in R^{cf}_{\mathbb{S}}$ and, thus, $(A \cup \{b\}) \notin \mathit{cf}(F^{cf}_{\mathbb{S}})$. Finally, since $(A \cup \{b\}) \subseteq (A \cup B) \subseteq S$ and $\mathit{cf}(F^{cf}_{\mathbb{S}})$ is downward-closed, this implies $S \notin \mathit{cf}(F^{cf}_{\mathbb{S}})$ which is a contradiction with the assumption that $S \in \mathit{cf}(F^{cf}_{\mathbb{S}})$. Hence, it must be that $\mathbb{A} = \mathbb{T}$ and, thus, that $A = S$ holds. Since $A \in \mathbb{S}$ by construction, this implies $S \in \mathbb{S}$. □

Lemma 6. (restated) Let $\mathbb{S}$ be a set-conflict-sensitive signature that contains $\emptyset$, then $\mathbb{S} \subseteq \mathit{dcl}(\mathbb{S}) \subseteq \mathit{adm}(F^{cf}_{\mathbb{S}})$.

Proof. Pick any set $S \in \mathit{dcl}(\mathbb{S})$, any argument $a \in S$ and any attack $(S', a) \in R^{cf}$. Then, $(S \cup S') \notin \mathit{dcl}(\mathbb{S})$ and, since $S, S' \in \mathbb{S}$ and $\mathbb{S}$ is conflict-sensitive, it follows that there is some $b \in S$ such that $(S \cup \{b\}) \notin \mathit{dcl}(\mathbb{S})$. This implies that $(S, b) \in R^{cf}$ and, thus, that S defends a against (S', a). Hence, S defends itself against all attacks in R^{cf}. □

Lemma 7. (restated) For every extension-set $\mathbb{S}$ that is set-conflict-sensitive and contains $\emptyset$, we have that $\mathit{adm}(F^{ucl}_{\mathbb{S}}) \supseteq \mathit{ucl}(\mathbb{S})$.

Proof. We show that $\mathit{adm}(F^{ucl}) \supseteq \mathit{ucl}(\mathbb{S})$. Let us first show that $\mathit{adm}(F^{ucl}) \supseteq \mathbb{S}$. Pick any $A \in \mathbb{S}$, $a \in A$ and $(S, a) \in R^{ucl}$. Then, by construction, we have that $S = \{x_a\}$ and, since $a \in A$, that $(A, x_a) \in R^{ucl}$, so A also defends against all attacks in R^{ucl}. Hence, we have that $\mathit{adm}(F^{ucl}) \supseteq \mathbb{S}$ hold. Pick now $A, B \in \mathbb{S}$. Then, $A, B \in \mathit{adm}(F^{ucl})$ and, thus, $A \cup B$ defends itself w.r.t. F^{ucl}. Furthermore, by construction, there are no attacks between elements of $A \cup B$ and thus $A \cup B \in \mathit{adm}(F^{ucl})$. □

The following lemma completes the proof of Proposition 4.

Lemma 8. (restated) For every extension-set $\mathbb{S}$ that is set-conflict-sensitive and contains $\emptyset$ we have $\mathit{adm}(F^{adm}) = \mathbb{S}$.

Proof. We consider the SETAF F^{adm} (see Definition 10) and show that $adm(F^{adm}) \supseteq \mathbb{S}$. First, note that from Lemma 6, we have that $adm(F^{cf}) \supseteq \mathbb{S}$. Let us show that $adm(F') \supseteq \mathbb{S}$ also holds. Pick any $a \in S$ and $(S', a) \in R'$. Then, by construction, we have that $S' = \{x_a\}$ and, since $a \in S$, that $(S, x_a) \in R'$, so S also defends against all attacks in R'. Hence, we have that $adm(F^{cf}) \supseteq \mathbb{S}$ and $adm(F') \supseteq \mathbb{S}$ hold. From Lemma 18, this implies $adm(F^{adm}) \supseteq adm(F^{cf} \cup F') \supseteq \mathbb{S}$ follows.

The other way around, Let us show that $adm(F) \subseteq \mathbb{S}$ also holds. Pick any $S \in adm(F)$. Then, for every argument $a \in S$ (there is an attack $(\{x_a\}, a) \in R^{adm}$ and, so) there must be an attack $(T_a, \{x_a\}) \in R^{adm}$ with $T_a \subseteq S$. Furthermore, by construction, we also have that $T_a \in \mathbb{S}$ and $a \in T_a$. Let $\mathbb{T} = \{T_a \mid a \in S\} \subseteq \mathbb{S}$ and $C = \bigcup \mathbb{T}$. Then, we have that $C = S$ and, from Lemma 5 and the fact that $S \in adm(F^{adm}_{\mathbb{S}}) \subseteq cf(F^{adm}_{\mathbb{S}}) \subseteq cf(F^{cf}_{\mathbb{S}})$, this implies that $S \in \mathbb{S}$. □

A.4 Signature of Complete Semantics

In this section we provide a detailed proof for Theorem 5.

Theorem 5. (restated) $\Sigma^{\infty}_{com} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \text{ is set-comp-closed and } \bigcap \mathbb{S} \in \mathbb{S}\}$

The following Lemma shows that the signature of every SETAF is indeed set-comp-closed and contains the intersection of all its elements.

Lemma 9. (restated) For every SETAF F we have that (a) the extension-set $com(F)$ is set-comp-closed and (b) $\bigcap com(F) \in com(F)$.

Proof. First, notice that $\bigcap com(F) = grd(F)$ and as the grounded extension is complete we obtain (b). In order to show (a) consider extension-sets $\mathbb{T}, \mathbb{U} \subseteq com(F)$ and sets $T = \bigcup \mathbb{T}$, $U = \bigcup \mathbb{U}$ such that $T, U \in dcl(com(F))$. From $T, U \in dcl(com(F))$, it follows that $T, U \in cf(F)$ and from $T = \bigcup \mathbb{T}$, $U = \bigcup \mathbb{U}$ with $\mathbb{T}, \mathbb{U} \subseteq com(F)$ it follows that T and U defend themselves. Hence, we get $T, U \in adm(F)$. If in addition we have $T \cup U \in cf(F)$, then by Lemma 2 we have that $T \cup U \in adm(F)$ and thus by Lemma 1 there is a unique $\subseteq$-minimal complete extension $E \in com(F)$ with $T \cup U \subseteq E$, i.e. $|\mathbb{C}_{com(F)}(T \cup U)| = 1$. If $T \cup U \notin cf(F)$ then there exists an attack $(S, a) \in R$ with $S \subseteq T \cup U$ and $a \in T \cup U$.

- If $a \in T$, as T is admissible, there is an attack (T', u) with $T' \subseteq T$ and $u \in S \setminus T \subseteq U$. Thus, $T \cup \{u\} \in PAtt_{com(F)}$.
- If $a \in U$, as U is admissible, there is an attack (U', s) with $U' \subseteq U$ and $s \in S \setminus U \subseteq T$. Now, as T is admissible, there is an attack (T', u) with $T' \subseteq T$ and $u \in U' \subseteq U$. Thus $T \cup \{u\} \in PAtt_{com(F)}$.

In both cases we have an $u \in U$ such that $T \cup \{u\} \in PAtt_{com(F)}$ and thus $com(F)$ is set-comp-closed. □

The realization corresponding Theorem 5 is based on the construction for the admissible semantics given in Definition 10. First, given an extension-set $\mathbb{S}$, by $reduced(\mathbb{S}) = \{S \setminus \bigcap \mathbb{S} \mid S \in \mathbb{S}\}$, we denote a reduced extension-set whose corresponding ground extension is empty. Furthermore, by $\mathit{admcl}(\mathbb{S}) = \{ \bigcup \mathbb{T} \mid \mathbb{T} \subseteq \mathbb{S}, \bigcup \mathbb{T} \in \mathit{dcl}(\mathbb{S}) \}$ we denote the admissible closure of $\mathbb{S}$, that is, the minimal extension-set that includes $\mathbb{S}$ and is admissible realizable.

Lemma 10. (restated) For every extension-set $\mathbb{S}$ that is set-comp-closed and satisfies $\bigcap \mathbb{S} \in \mathbb{S}$, we have that $\mathit{com}(F^{com}_{\mathbb{S}}) = \mathbb{S}$.

Proof. Consider the SETAF $F^{com}_{\mathbb{S}}$ from Definition 12. Let $\mathbb{S}' = \mathit{reduced}(\mathbb{S})$, $\mathbb{S}^* = \mathit{dcl}(\mathbb{S}') \cap \mathit{ucl}(\mathbb{S}')$ and let us show first that $\mathit{com}(F^{com}_{\mathbb{S}'}) = \mathbb{S}'$. Notice that $\mathbb{S}'$ is still set-comp-closed and $\emptyset \in \mathbb{S}'$.

Let us show that $\mathbb{S}^*$ is set-conflict-sensitive. Consider $T, U \in \mathbb{S}^*$ such that $T \cup U \notin \mathbb{S}^*$. Then, by construction of $\mathbb{S}^*$, we have that $T \cup U \notin \mathit{dcl}(\mathbb{S}')$ and thus $\mathbb{C}_{\mathit{com}(F)}(T \cup U) = \emptyset$. Now as $\mathbb{S}'$ is set-comp-closed there is an argument $u \in U$ such that $T \cup \{u\} \in \mathit{PAtt}_{\mathbb{S}}$. That is, for $T, U \in \mathbb{S}^*$ such that $T \cup U \notin \mathbb{S}^*$ there is an argument $u \in U$ such that $T \cup \{u\} \in \mathit{PAtt}_{\mathbb{S}}$.

Hence, the extension-set $\mathbb{S}^*$ is set-conflict-sensitive and, from Lemma 8, it follows that $\mathit{adm}(F^{adm}_{\mathbb{S}^*}) = \mathbb{S}^*$. Now consider the new attacks in R' and how they affect the admissibility of sets. Notice that only auxiliary arguments x_a are attacked and thus each set that is admissible in $F^{adm}_{\mathbb{S}}$ is admissible in $F^{com}_{\mathbb{S}^*}$ as well (Lemma 18). Hence, we have $\mathit{adm}(F^{adm}_{\mathbb{S}^*}) \subseteq \mathit{adm}(F^{com}_{\mathbb{S}'})$. Let us show $\mathbb{S}' \subseteq \mathit{com}(F^{com}_{\mathbb{S}'})$. Consider $S \in \mathbb{S}' \subseteq \mathbb{S}^*$. By the above, we have that $S \in \mathit{adm}(F^{adm}_{\mathbb{S}})$, and it remains to be shown that S does not defend any $a \in \mathit{Args}_{\mathbb{S}} \setminus S$, i.e., does not attack any x_a for $a \in \mathit{Args}_{\mathbb{S}} \setminus S$. By construction of $F^{com}_{\mathbb{S}'}$ the set S only attacks arguments x_a with $a \in S$ and thus $S \in \mathit{com}(F^{com}_{\mathbb{S}'})$ follows. The other way around, let us show $\mathbb{S}' \supseteq \mathit{com}(F^{com}_{\mathbb{S}'})$. Consider $S \in \mathit{adm}(F^{com}_{\mathbb{S}'})$. We next show that if $S \notin \mathbb{S}'$ then $S \notin \mathit{com}(F^{com}_{\mathbb{S}'})$. To this end we consider two cases.

- $S \in \mathit{adm}(F^{com}_{\mathbb{S}}) \setminus \mathbb{S}^*$: Notice that $\mathbb{S}^* = \mathit{adm}(F^{adm}_{\mathbb{S}^*})$. Consider a set S that is admissible in $F^{com}_{\mathbb{S}'}$ but not in $F^{adm}_{\mathbb{S}^*}$. This can only be because of the attacks introduced with R'. That is, there is some x_s with $s \in S$ that prevents that S is admissible in $F^{adm}_{\mathbb{S}^*}$ and an attack $(A \cup B, x_s) \in R'$ with which S defends itself against x_s in $F^{com}_{\mathbb{S}'}$. That is $A, B \subseteq S$ and, by the definition of R', we have that there is a unique completion $C = \mathsf{C}_{\mathbb{S}'}(A \cup B)$ and $s \in C$ (recall that in R' we only draw attacks for $A \cup B$ with a unique completion). As $C \in \mathbb{S}' \subseteq \mathbb{S}^*$ and $s \in C$, by construction, there is an attack (C, x_s) in $F^{adm}_{\mathbb{S}^*}$ That is, if $C \subseteq S$ then S attacks x_s in $F^{adm}_{\mathbb{S}^*}$, a contradiction to our initial assumption. Hence we have $C \not\subseteq S$. But then we can argue as before that, in $F_{\mathbb{S}'}$, S defends all arguments in C and thus S is not complete.

- $S \in \mathbb{S}^* \setminus \mathbb{S}'$: Then, there is a set $\mathbb{T} \subseteq \mathbb{S}'$ such that $\bigcup \mathbb{T} = S$ and $S \in \mathit{dcl}(\mathbb{S}')$. As $\mathbb{S}'$ is set-comp-closed for each $A, B \in \mathbb{S}'$ with $A, B \subseteq S$ we have a unique completion set $\mathsf{C}_{\mathbb{S}'}(A \cup B)$ (as $A \cup B \in \mathit{dcl}(\mathbb{S}')$ we cannot have a conflict in $A \cup B$). Towards a contradiction assume that for all $A, B \in \mathbb{S}'$ such that $A, B \subseteq S$ we have $\mathsf{C}_{\mathbb{S}'}(A \cup B) \subseteq S$. Then we can iteratively replace $A, B \in \mathbb{T}$ by $\mathsf{C}_{\mathbb{S}'}(A \cup B)$ and we end up with a single set in $\mathbb{T}$. But then $S \in \mathbb{S}'$, a contradiction. Thus there are two sets $A, B \in \mathbb{S}'$ such that $A, B \subseteq S$ and $A \cup B \notin \mathbb{S}'$, and there is also a unique set $C \in \mathsf{C}_{\mathbb{S}'}(A \cup B)$ with $C \not\subseteq S$ Let $c \in C \setminus S$, we next argue the S defends c and thus is not complete. By construction $A \cup B$ (and thus S) attacks all x_a with $a \in C$. Now consider a set set $D \in \mathbb{S}^*$ that attacks c. We have that $(A \cup B) \cup D \notin \mathit{dcl}(\mathbb{S}')$ and thus there is an $d' \in D$ such that $(A \cup B) \cup \{d'\} \in \mathit{PAtt}(\mathbb{S}^*) = \mathit{PAtt}(\mathbb{S}')$. That is $A \cup D$ defends c against both possible kinds of attackers and thus defends c.

Combining the both cases we obtain that if $S \notin \mathbb{S}'$ then $S \notin com(F^{com}_{\mathbb{S}'})$. Finally, just note that $com(F^{com}_{\mathbb{S}})$ just adds to $com(F^{com}_{\mathbb{S}'})$ the arguments in $\bigcup \mathbb{S}$ as unstacked, Hence, $S \in com(F^{com}_{\mathbb{S}})$ iff $(S \setminus \bigcap \mathbb{S}) \in com(F^{com}_{\mathbb{S}'})$ and, thus, $com(F^{com}_{\mathbb{S}}) = \mathbb{S}$.

Given a set-comp-closed $\mathbb{S}$ with $\bigcap \mathbb{S} \in \mathbb{S}$. We first construct an AF for $\mathbb{S}' = \{S \setminus \bigcap \mathbb{S} \mid S \in \mathbb{S}\}$ and then add the arguments of $\bigcap \mathbb{S} \in \mathbb{S}$ as isolated arguments. Now consider $\mathbb{S}^* = \{\bigcup \mathbb{T} \mid \mathbb{T} \subseteq \mathbb{S}', \bigcup \mathbb{T} \in dcl(\mathbb{S}')\}$ and constructed the SETAF F^{adm} for $\mathbb{S}^*$. We now expand the SETAF F^{adm} by the following attacks

$$R' = \{(A \cup B, x_a) \mid A, B \in \mathbb{S} \setminus \{\emptyset\}, a \in \mathsf{C}_{\mathbb{S}'}(A \cup B)\}$$

For $F^{com} = (A^{com}, R^{com}) = (A^{adm}, R^{adm} \cup R')$ we then have $com(F^{com}) = \mathbb{S}'$. We first show that $\mathbb{S}^*$ is admissible realizable by F^{adm}. First, by construction of $\mathbb{S}'$, we have $\emptyset \in \mathbb{S}^*$. It remains to show that $\mathbb{S}^*$ is also set-conflict-sensitive. Consider $T, U \in \mathbb{S}^*$ such that $T \cup U \notin \mathbb{S}^*$. Then, by construction of $\mathbb{S}^*$, we have that $T \cup U \notin dcl(\mathbb{S}')$ and thus $\mathbb{C}_{com(F)}(T \cup U) = \emptyset$. Now as $\mathbb{S}'$ is set-comp-closed there is an argument $u \in U$ such that $T \cup \{u\} \in PAtt_{\mathbb{S}}$. That is, for $T, U \in \mathbb{S}^*$ such that $T \cup U \notin \mathbb{S}^*$ there is an argument $u \in U$ such that $T \cup \{u\} \in PAtt_{\mathbb{S}}$. Hence, $\mathbb{S}^*$ is set-conflict-sensitive. By Theorem 4 we have that $\mathbb{S}^* = adm(F^{adm})$. Now consider the new attacks in R' and how they affect the admissibility of sets. Notice that only auxiliary arguments x_a are attacked and thus each set that is admissible in F^{adm} is admissible in F^{com} as well, i.e. $adm(F^{adm}) \subseteq adm(F^{com})$. $\mathbb{S}' \subseteq com(F^{com})$: Consider $S \in \mathbb{S}' \subseteq \mathbb{S}^*$. By the above we have $S \in adm(F^{com})$ it remains to show that S does not defend any $a \in Args_{\mathbb{S}} \setminus S$, i.e., does not attack any x_a for $a \in Args_{\mathbb{S}} \setminus S$. By construction of F^{com} the Set S only attacks arguments x_a with $a \in S$ and thus $S \in com(F^{com})$ follows. $\mathbb{S}' \supseteq com(F^{com})$: Now let us consider $S \in adm(F^{com})$. We next show that if $S \notin \mathbb{S}'$ then $S \notin com(F^{com})$. To this end we consider two cases.

- $S \in \mathbb{S}^* \setminus \mathbb{S}'$: Then S contains two sets $A, B \in \mathbb{S}'$ such that $A \cup B \notin \mathbb{S}'$ and there is a unique set $C \in \mathsf{C}_{\mathbb{S}'}(A \cup B)$ with $C \not\subseteq S$. Let $c \in C \setminus S$, we next argue the S defends c and thus is not complete. By construction $A \cup B$ (and thus S) attacks all x_a with $a \in C$. Now consider a set set $D \in \mathbb{S}^*$ that attacks c. We have that $(A \cup B) \cup D \notin dcl(\mathbb{S}')$ and thus there is an $d' \in D$ such that $(A \cup B) \cup \{d'\} \in PAtt(\mathbb{S}^*) = PAtt(\mathbb{S}')$. That is $A \cup D$ defends c against both possible kinds of attackers and thus defends c.

- $S \in adm(F^{com}) \setminus \mathbb{S}^*$: Notice that $\mathbb{S}^* = adm(F^{adm})$. Consider a set S that is admissible in F^{com} but not in F^{adm}. This can only be because of the attacks introduce with R'. That is, S contains $A, B \in \mathbb{S}'$ and there is a unique set $C \in \mathsf{C}_{\mathbb{S}'}(A \cup B)$ with $C \not\subseteq S$. But then we can argue as before that S defends C and thus is not complete.

Combining the both cases we obtain that if $S \notin \mathbb{S}'$ then $S \notin com(F^{com})$. Finally, by adding the arguments $\bigcap \mathbb{S}$ to F^{com} we obtain our realization of $\mathbb{S}$, i.e. $com(A^{com} \cup \bigcap \mathbb{S}, R^{com}) = \mathbb{S}$. $\square$

B Proofs of Section 4

B.1 Signatures for Conflict-Free and Naive Semantics

The *Proof of Theorem 6* is by the following two propositions.

Proposition 8. $\Sigma^k_{cf} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S}$ *is downward-closed and* k*-tight*$\}$

Proof. First we show that $cf(F)$ is downward-closed and k-tight for every k-SETAF F. If $S \in cf(F)$ than no subset of S can contain a conflict as then S would contain that conflict as well, i.e. all subsets are conflict-free as well and thus $cf(F)$ is downward-closed. Now consider an argument a such that $S \cup \{a\} \notin cf(F)$. Then $S \cup \{a\}$ attacks $S \cup \{a\}$, that is either there is a set $S' \subseteq S \cup \{a\}$ with $(S', a) \in R$ or there is a set E with $a \in E$ and $(E, b) \in R$ for some $b \in S$. In the former case $S' \cup \{a\}$ is of size $\leq k+1$ and $S' \cup \{a\} \notin dcl(cf(F))$. In the latter case consider $S' = (E \setminus \{a\}) \cup \{b\}$ which is of size $\leq k+1$ and $S' \cup \{a\} = E \cup \{b\} \notin dcl(cf(F))$. In both cases we have $S' \cup \{a\} \in PAtt^k_{cf(F)}$ and thus the condition for $cf(F)$ being tight is satisfied.

Now consider $F^{cf,k}_{\mathbb{S}}$ from Definition 14. We have that $cf(F^{cf,k}_{\mathbb{S}}) = \mathbb{S}$.
1) Let us show first that $cf(F^{cf,k}_{\mathbb{S}}) \supseteq \mathbb{S}$. Pick any $S \in \mathbb{S}$ and any attack $(S', a) \in R$ with $S' \subseteq S$. By construction, we have that $(S' \cup \{a\}) \notin dcl(\mathbb{S})$ and, thus, that $(S' \cup \{a\}) \not\subseteq S$. Hence, since $S' \subseteq S$, it follows that $a \notin S$ and that S is conflict-free. Hence, we have that $cf(F^{cf,k}_{\mathbb{S}}) \supseteq \mathbb{S}$.
2) Now consider $cf(F^{cf,k}_{\mathbb{S}}) \supseteq \mathbb{S}$. Pick $S \subseteq Args_{\mathbb{S}}$ with $S \notin \mathbb{S}$ as S is downward closed we have that there is an $S' \in \mathbb{S}$ such that $S' \subseteq S$, and w.l.o.g. assume that S' is a maximal such set. Pick $a \in S \setminus S'$. As $\mathbb{S}$ is k-tight there is an attack $(B, a) \in R$ for some $B \subseteq S'$ and thus $S \notin cf(F^{cf,k}_{\mathbb{S}})$. □

Proposition 9. $\Sigma^k_{naive} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S}$ *is incomparable and* $dcl(\mathbb{S})$ *is* k*-tight*$\}$

Proof. First consider $naive(F)$ for some k-SETAF F. By definition $naive(F)$ is incomparable and $dcl(naive(F)) = cf(F)$. Thus by Proposition 8 $dcl(naive(F))$ is k-tight

To realize a set $\mathbb{S}$ that is incomparable and such that $dcl(\mathbb{S})$ is k-tight consider $\mathbb{S}' = dcl(\mathbb{S})$ and realize it by the construction of Proposition 8. Let F be the resulting SETAF. Then we have that $cf(F) = \mathbb{S}'$ and by construction $\mathbb{S}$ contains exactly the $\subset$-maximal elements of $\mathbb{S}'$. Hence, $naive(F) = \mathbb{S}$. □

B.2 Signatures for Stable Semantics

Theorem 7. (restated) $\Sigma^k_{stb} = \{\mathbb{S} \mid \mathbb{S}$ is incomparable and k-tight$\}$

Proof. First consider $stb(F)$ for some k-SETAF F. As $stb(F) \subseteq pref(F)$ it is incomparable. Now consider $E \in stb(F)$. By definition for each argument $a \notin E$ there is an attack (B, a) with $|B| \leq k$. That is there is no $E' \in stb(F)$ with $B \cup \{a\} \subseteq E'$ and thus $B \cup \{a\} \in PAtt_{stbF}$. That is $stb(F)$ is also k-tight.

Now consider the SETAF F^{stb}_k from Definition 15. We have that $stb(F) = \mathbb{S}$.
1) Let us show first that $stb(F^{stb}_k) \supseteq \mathbb{S}$. Pick any $S \in \mathbb{S}$ and any attack $(S', a) \in R$ with $S' \subseteq S$.

By construction, we have that $(S' \cup \{a\}) \in \mathit{PAtt}_{\mathbb{S}}$ and, thus, that $(S' \cup \{a\}) \not\subseteq S$. Now consider $a \in \mathit{Args}_{\mathbb{S}} \setminus S$. As $\mathbb{S}$ is k-tight there exists $B \subseteq S, |B| \leq k$ such that $B \cup \{a\} \in \mathit{PAtt}_{\mathbb{S}}$ and thus $(B, a) \in R$. Finally, consider $x_E \in \{x_S \mid S \notin \mathbb{S}$ and S $\subseteq$-maximal in $\mathit{dcl}(\mathbb{S})\}$. We have that S and E are incomparable and thus there is an argument $a \in E$ such that $(\{a\}, x_E) \in R$. That is, $S \in \mathit{stb}(F_k^{\mathit{stb}})$.
2) Now consider $\mathit{stb}(F_k^{\mathit{stb}}) \supseteq \mathbb{S}$. Pick $S \subseteq \mathit{Args}_{\mathbb{S}}$. If $S \notin \mathit{naive}(F_k^{\mathit{stb}})$ then it is not stable, thus we will assume $S \in \mathit{naive}(F_k^{\mathit{stb}})$. Notice that by construction we have $\mathit{naive}(F) = \{x_S \mid S$ $\subseteq$-maximal in $\mathit{dcl}(\mathbb{S})\}$. That is, there is an argument $x_S \in A$ with $x_S \notin S$ and S not attacking x_S. Thus, $S \notin \mathit{stb}(F_k^{\mathit{stb}})$. □

B.3 Signatures of Admissible and Preferred Semantics

Remark 3. Given an extension-set $\mathbb{S}$, if there exists a set P that meets the conditions of Definition 16 one such set can be computed by a fixed point iteration as follows. In an initial phase, for each $S \in \mathbb{S}$ consider all subsets A_S^i of size $\min(k, |S|)$ and for $b \in \mathit{Args}_{\mathbb{S}} \setminus S$ and add (A_S^i, b) to P. Then iteratively check condition (ii) and remove attacks that violate the condition from P. When the fixed point is reached, i.e. P satisfies (ii), check condition (i). If (i) is valid we have found a set P that meets the conditions of Definition 16, otherwise there is no such set.

In this section we provide a full *proof for Theorem 8.*

Theorem 8. (restated) $\Sigma_{\mathit{adm}}^k = \{\mathbb{S} \neq \emptyset \mid \mathbb{S}$ is k-defensive and contains $\emptyset\}$ and $\Sigma_{\mathit{pref}}^k = \{\mathbb{S} \neq \emptyset \mid \mathbb{S}$ is incomparable and k-defensive$\}$

We first show the result for admissible semantics.

Proposition 10. $\Sigma_{\mathit{adm}}^k = \{\mathbb{S} \neq \emptyset \mid \mathbb{S}$ *is k-defensive and contains* $\emptyset\}$

Lemma 11. (restated) For any SETAF F we have that $\mathit{adm}(F)$ is k-defensive and contains $\emptyset$.

Proof. First, notice that the empty set is always admissible. Now we consider the set $P = \{(S, a) \in R \mid S \subseteq \mathit{Args}_{\mathit{adm}}(F)\}$ and show that it satisfies the two conditions for $\mathbb{S}$ being k-defensive.
1) Assume there are two admissible sets A, B such that the set $C = A \cup B$ is not admissible. By Lemma 2 the set C defends itself against all attackers and thus there must be a conflict in C, i.e. there exists an attack $(S, a) \in R$ with $S \subseteq C$ and $a \in A$.

- If $a \in A$ then, as A is conflict-free, $S \cap B \neq \emptyset$. Moreover, as A is admissible it has to defend itself against (A, a), i.e. there is an attack (S', b) with $S' \subseteq A$ and $b \in S \cap B$. Hence, we have $(S', b) \in P$.

- If $a \in B$ then, as B is conflict-free, $S \cap A \neq \emptyset$. Moreover, as B is admissible it has to defend itself against (S, a), i.e. there is an attack (S^*, c) with $S^* \subseteq B$ and $c \in S \cap A$. Now, as A is admissible as well, there is also an attack (S', b) with $S' \subseteq E$ and $b \in S' \subseteq B$. Hence, we have $(S', b) \in P$.

2) If $(S, b) \in P$ then S attacks b in F. Thus each set $S' \in adm(F)$ with $b \in S'$ defends itself against S, i.e. for each $S' \in adm(F)$ with $b \in S'$ there is a pair $(A_s^j, a) \in R$ with $A_s^i \subseteq S'$ and $a \in S$. Thus, also $(A_s^j, a) \in P$ and the second condition is satisfied.

We obtain that $adm(F)$ is set-conflict-sensitive. □

Lemma 12. (restated) Let $\mathbb{S}$ be a k-defensive signature that contains $\emptyset$, P a set that meets the conditions of Definition 16 and $S \subseteq Args_{\mathbb{S}}$ be some set of arguments such that $S = \bigcup \mathbb{T}$ for some subset $\mathbb{T} \subseteq \mathbb{S}$. Then, we have that $S \in cf(F_{\mathbb{S},P}^{cf})$ implies $S \in \mathbb{S}$.

Proof. Consider a SETAF $F_{\mathbb{S},P}^{cf}$ as given in Definition 17. Let us define $\mathbb{A} \subseteq \mathbb{T}$ such that $\bigcup \mathbb{A} \in \mathbb{S}$ and there is no $\mathbb{A}' \subseteq \mathbb{T}$ such that $\mathbb{A} \subset \mathbb{A}'$ and $\bigcup \mathbb{A}' \in \mathbb{S}$. Note that such $\mathbb{A}$ always exists because $\bigcup \emptyset = \emptyset \in \mathbb{S}$. We also define $A = \bigcup \mathbb{A}$. Towards a contradiction assume $\mathbb{A} \subset \mathbb{T}$ and pick any $B \in \mathbb{T} \backslash \mathbb{A}$. Then, by construction, we have that $A, B \in \mathbb{S}$, $(A \cup B) \subseteq S$ and that $(A \cup B) \notin \mathbb{S}$. Furthermore, since $\mathbb{S}$ is k-defensive, it follows that there are $A' \subseteq A$ and $b \in B$ such that $|A'| \leq k$ and $(A', b) \in P$. This implies that there is an attack $(A', b) \in R_{\mathbb{S},P}^{cf}$ and, thus, $(A' \cup \{b\}) \notin cf(F_{\mathbb{S},P}^{cf})$. Finally, since $(A' \cup \{b\}) \subseteq (A \cup B) \subseteq S$ and $cf(F_{\mathbb{S},P}^{cf})$ is downward-closed, this implies $S \notin cf(F_{\mathbb{S},P}^{cf})$ which is a contradiction with the assumption that $S \in cf(F_{\mathbb{S},P}^{cf})$. Hence, it must be that $\mathbb{A} = \mathbb{T}$ and, thus, that $A = S$ holds. Since $A \in \mathbb{S}$ holds by construction, this implies $S \in \mathbb{S}$. □

Lemma 13. (restated) Let $\mathbb{S}$ be a k-defensive signature that contains $\emptyset$, Then $\mathbb{S} \subseteq dcl(\mathbb{S}) \subseteq adm(F_{\mathbb{S},P}^{cf})$.

Proof. Pick any set $S \in dcl(\mathbb{S})$, any argument $a \in S$ and any attack $(S', a) \in R_{\mathbb{S},P}^{cf}$. Then, $(S \cup S') \notin dcl(\mathbb{S})$ and, since $S, S' \in \mathbb{S}$ and $\mathbb{S}$ is k-defensive it follows that there are some $S^* \subseteq S$, $b \in S'$ such that $(S^*, b) \in P$. This implies that $(S^*, b) \in R_{\mathbb{S},P}^{cf}$ and, thus, that S defends a against (S', a). Hence, S defends itself against all attacks in $R_{\mathbb{S},P}^{cf}$. □

Lemma 15. (restated) For every extension-set $\mathbb{S}$ that contains $\emptyset$ we have that $S \in adm(F_{\mathbb{S}}^{def})$ iff $S = \bigcup \mathbb{T}$ for some $\mathbb{T} \subseteq \mathbb{S}$.

Proof. First notice that there are no conflicts between arguments in $Args_{\mathbb{S}}$ and all arguments not in $Args_{\mathbb{S}}$ are self-attacking. It thus suffices to show that S defends itself in $F_{\mathbb{S}}^{def}$ iff $S = \bigcup \mathbb{T}$ for some $\mathbb{T} \subseteq \mathbb{S}$.

$\Rightarrow$: Let $S \in adm(F_{\mathbb{S}}^{def})$ and consider an argument $a \in S$. S attacks all the arguments $\alpha_{a\gamma}$ that attack a and by construction this implies that S contains a model M of $\mathcal{CD}_a^{\mathbb{S}}$. By Lemma 14 we have $M \cup \{a\} \in \mathbb{S}$. As this holds for each argument $a \in S$ there is a $\mathbb{T} \subseteq \mathbb{S}$ such that $S = \bigcup \mathbb{T}$.

$\Leftarrow$: Let $\mathbb{T} \subseteq \mathbb{S}$ and $S = \bigcup \mathbb{T}$. Consider $a \in S$ and a set $T \in \mathbb{T}$ corresponding with $a \in T$. By Lemma 14 we have that $T \setminus \{a\}$ is a model of $\mathcal{CD}_a^{\mathbb{S}}$ and thus attacks all of the arguments $\alpha_{a\gamma}$. That is a is defended by S. Hence, $S \in adm(F_{\mathbb{S}}^{def}$. □

Lemma 16. (restated) For every extension-set $\mathbb{S}$ that is k-defensive and contains $\emptyset$ we have that $adm(F_{\mathbb{S},P}^{adm}) = \mathbb{S}$.

Proof. We consider the k-SETAF $F^{adm}_{\mathbb{S},P} = F^{cf}_{\mathbb{S},P} \cup F^{def}_{\mathbb{S}}$ from Definition 19. and show that $\mathbb{S} = adm(F^{adm}_{\mathbb{S},P})$.

- $\mathbb{S} \subseteq adm(F^{adm}_{\mathbb{S},P})$: We have that, by Lemma 13 $\mathbb{S} \subseteq adm(F^{cf}_{\mathbb{S},P})$ and, by Lemma 15, $\mathbb{S} \subseteq adm(F^{def}_{\mathbb{S}})$. Hence, by Lemma 18, $\mathbb{S} \subseteq adm(F^{adm}_{\mathbb{S},P})$.
- $\mathbb{S} \supseteq adm(F^{adm}_{\mathbb{S},P})$: Consider $S \in adm(F^{adm}_{\mathbb{S},P})$ which by definition is conflict-free in $F^{adm}_{\mathbb{S},P}$. Notice that, attacks from $F^{cf}_{\mathbb{S},P}$ cannot be used to defend against attacks from $F^{def}_{\mathbb{S}}$ and vice versa. Thus, by Lemma 17 and the above observation, $S \in adm(F^{cf}_{\mathbb{S},P})$ and $S \in adm(F^{def}_{\mathbb{S}})$. By Lemma 15 we have that $S = \bigcup \mathbb{T}$ for some $\mathbb{T} \subseteq \mathbb{S}$. Now by Lemma 12 we have that if such an S is in $cf(F^{cf}_{\mathbb{S},P})$ then $S \in \mathbb{S}$. As we already know that $S \in adm(F^{cf}_{\mathbb{S},P}) \subseteq \mathbb{S}$ we obtain $S \in \mathbb{S}$.

□

Finally we consider the signature for preferred semantics.

Proposition 11. $\Sigma^k_{pref} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S}$ *is incomparable and k-defensive*$\}$

Proof. First consider the set $pref(F)$ for an arbitrary SETAF $F = (A, R)$. The extension-set $pref(F)$ is incomparable by the definition of preferred semantics.

Now we consider the set $P = \{(S, a) \in R \mid S \subseteq Args_{adm}(F)\}$ and show that it satisfies the two conditions for $\mathbb{S}$ being k-defensive.

1) Consider arbitrary extensions $E, T \in pref(F)$ with $E \neq T$. By the maximality of E and T we have that $E \cup T \notin pref(F)$, $E \cup T$ is not contained in any preferred extension, and, by Lemma 2, we know that $E \cup T$ defends itself against all attackers.

That is, there is a conflict $(B, a) \in R$ such that $B \subseteq E \cup T$ and $a \in E \cup T$.

- If $a \in E$ then, as E is conflict-free, $B \cap T \neq \emptyset$. Moreover, as E is admissible it has to defend itself against (B, a), i.e. there is an attack (S, b) with $S \subseteq E$ and $b \in B \cap T$. Hence, we have $(S, b) \in P$.
- If $a \in T$ then, as T is conflict-free, $B \cap E \neq \emptyset$. Moreover, as T is admissible it has to defend itself against (B, a), i.e. there is an attack (S', c) with $S' \subseteq T$ and $c \in B \cap E$. Now, as E is admissible as well, there is also an attack (S, b) with $S \subseteq E$ and $b \in S \subseteq T$. Hence, we have $(S, b) \in P$.

2) If $(S, b) \in P$ then S attacks b in F. Thus each set $S' \in pref(F)$ with $b \in S'$ defends itself against S, i.e. for each $S' \in pref(F)$ with $b \in S'$ there is a pair $(A^j_s, a) \in R$ with $A^i_s \subseteq S'$ and $a \in S$. Thus, also $(A^j_s, a) \in P$ and the second condition is satisfied.

We obtain that $pref(F)$ is k-defensive.

Now consider an extension-set $\mathbb{S}$ that is incomparable and k-conflict-sensitive. The set $\mathbb{S}' = \mathbb{S} \cup \{\emptyset\}$ is k-conflict-sensitive and contains the empty set. By Proposition 10 there is a k-SETH F such that $adm(F) = \mathbb{S}'$. As the preferred extensions are the $\subseteq$-maximal admissible sets we also have $pref(F) = \mathbb{S}$. □

Proposition 12. *Every k-tight incomparable extension-set is also k-defensive.*

Proof. Consider some k-tight incomparable extension-set $\mathbb{S}$. We define P as the set of pairs (S', a) with $S' \cup \{a\} \in \mathit{PAtt}_{\mathbb{S}}^k$ and $S' \subseteq S \in \mathbb{S}$. We next show that P meets the conditions of Definition 16.

- condition (i): Consider $S, T \in \mathbb{S}$ with $S \cup T \notin \mathbb{S}$. Then there is a $t \in T \setminus S$ such that $S \cup \{t\} \notin \mathbb{S}$ and as $\mathbb{S}$ is k-tight there is a set $S' \subseteq S$ with $S' \cup \{t\} \in \mathit{PAtt}_{\mathbb{S}}^k$. Thus $(S', t) \in P$ and the condition is satisfied

- condition (ii): Now consider a set $T \in \mathbb{S}$ that is attacked by $(S', t) \in P$, i.e. $t \in T$. We have that S' must contain an argument s such that $T \cup \{s\} \notin \mathbb{S}$ otherwise, as $\mathbb{S}$ is incomparable, $S' \cup \{t\} \subseteq T$ and thus $S' \cup \{t\} \notin \mathit{PAtt}_{\mathbb{S}}^k$. Then as $\mathbb{S}$ is tight there is a pair $(T', s) \in P$ with $T' \subseteq T$ and hence also this condition is satisfied.

As the constructed P satisfies both conditions of Definition 16 we obtain that $\mathbb{S}$ is k-defensive. □

Example 9. Reconsider the argument set $A = \{a_1, a_2, \dots, a_{k+1}, a_{k+2}\}$ and the extension-set $\mathbb{T} = \{S \subseteq A \mid |S| = k+1\}$ from Example 7. We next argue that the extension set $\mathbb{T}$ is not k-defensive. For the sets $S_1 = \{a_1, a_2, \dots, a_{k+1}\}$ and $S_2 = \{a_2, a_2, \dots, a_{k+2}\}$, we need $S' \subset S_1$ and $t \in \mathsf{S}_2$ such that $S' \cup \{t\} \in \mathit{PAtt}_{\mathbb{S}}^k$. Indeed the only option for t is $a_k + 2$ as otherwise $S' \cup \{t\} \subseteq S$. But we also have that for any $S' \subset S$ the set $S \cup \{a_{k+2}\}$ is contained in some $T \in \mathbb{T}$ and thus $S \cup \{a_{k+2}\} \notin \mathit{PAtt}_{\mathbb{T}}^k$.

Hence, $\mathbb{T}$ cannot be realized as preferred extensions/admissible sets of a k-SETAF. However, one can easily verify that $\mathbb{T}$ is $(k+1)$-defensive and thus $\mathbb{T}$ can be realized as preferred extensions of a $(k+1)$-SETAF as well as $\mathbb{T} \cup \{\emptyset\}$ can be realized as admissible extensions of a $(k+1)$-SETAF. ◊

Note that, for incomparable $\mathbb{S}$, whenever $\mathbb{S}$ is k-tight, also $\mathbb{S}$ is k-defensive. Hence, for k-SETAFs, the preferred semantics is more expressible than the semantics semantics. We next show that preferred semantics is indeed strictly more expressive than stable semantics.

Example 10. We consider the argument set $A = B \cup C \cup \{e\}$ with $B = \{b_1, b_2, \dots, b_{k+1}\}$, $C = \{c_1, c_2, \dots, c_{k+1}\}$ and the extension-set $\mathbb{S}$ that contains (i) the set B, and (ii) the sets $B \cup \{c_i, e\} \setminus \{b_i\}$ for $1 \leq i \leq k+1$. It is easy to verify that $\mathbb{S}$ is incomparable. We next argue that the set $\mathbb{S}$ is not k-tight. Consider $B \in \mathbb{S}$ and the argument e. We have that $S \cup \{e\} \notin \mathbb{S}$ but for each $S' \subset S$ with $|S'| \leq k$ the set $S' \cup \{e\}$ is contained in one of the sets in $\mathbb{S}$ and thus $S' \cup \{e\} \notin \mathit{PAtt}_{\mathbb{S}}^k$. That is, $\mathbb{S}$ is not tight and can not be realized with a k-SETAF under stable semantics. However, one can easily verify that $\mathbb{S}$ is conflict-sensitive and thus $\mathbb{S}$ can be realized with a 1-SETAF (and more general with a k-SETAF for any $k \geq 1$) under preferred semantics. ◊