DBAI

TECHNICAL REPORT

INSTITUT FÜR INFORMATIONSSYSTEME

ABTEILUNG DATENBANKEN UND ARTIFICIAL INTELLIGENCE

Technical Note: Complexity of Stage Semantics in Argumentation Frameworks

DBAI-TR-2009-66

Wolfgang Dvořák **Stefan Woltran**

DBAI TECHNICAL REPORT

2009

Institut für Informationssysteme
Abteilung Datenbanken und
Artificial Intelligence
Technische Universität Wien
Favoritenstr. 9
A-1040 Vienna, Austria
Tel: +43-1-58801-18403
Fax: +43-1-58801-18492
sekret@dbai.tuwien.ac.at
www.dbai.tuwien.ac.at

TECHNISCHE UNIVERSITÄT WIEN

DBAI TECHNICAL REPORT

DBAI TECHNICAL REPORT DBAI-TR-2009-66, 2009

Technical Note: Complexity of Stage Semantics in Argumentation Frameworks

Wolfgang Dvořák[1] **Stefan Woltran**[2]

Abstract. In this work, we answer two questions about the complexity of semi-stable semantics for abstract argumentation frameworks raised by Dunne and Caminada (2008): we show Π_2^P-completeness for the problem of deciding whether an argument is skeptically accepted, and respectively, Σ_2^P-completeness for the problem of deciding whether an argument is credulously accepted under the semi-stable semantics. Furthermore, we extend these complexity bounds to the according decision problems for stage semantics as introduced by Verheij (1997). We also discuss two approaches towards tractability: first, we prove that the problems under consideration are fixed-parameter tractable with respect to tree-width; second, we show that the problems remain intractable when considering frameworks of bounded cycle-rank.

[1]Institute for Information Systems 184/2, Technische Universität Wien, Favoritenstrasse 9-11, 1040 Vienna, Austria. E-mail: dvorak@dbai.tuwien.ac.at

[2]Institute for Information Systems 184/2, Technische Universität Wien, Favoritenstrasse 9-11, 1040 Vienna, Austria. E-mail: woltran@dbai.tuwien.ac.at

1 Introduction

In Artificial Intelligence (AI), the area of argumentation [2] has become one of the central issues during the last years. Argumentation provides a formal treatment for reasoning problems arising in a number of interesting applications fields, including Multi-Agent Systems and Law Research. In a nutshell, so-called abstract argumentation frameworks (AFs) formalize statements together with a relation denoting rebuttals between them, such that the semantics gives an abstract handle to solve the inherent conflicts between statements by selecting admissible subsets of them. Several such semantics have already been proposed by Dung in his seminal paper [6], but there are several others which received significant interest lately.

One such approach is known as stage semantics and was introduced by Verheij [14] more than ten years ago. With the work on semi-stable semantics by Caminada [3], who revived Verheij's basic concepts, stage semantics are nowadays mentioned as an important alternative (see, e.g. [1]) to Dung's original semantics. The underlying idea of stage semantics is to maximize not only the arguments included in an extension but also those attacked by such an extension.

In this work, we give exact complexity bounds for typical decision problems assigned to argumentation frameworks. In particular, we prove Σ_2^P-hardness, and resp. Π_2^P-hardness, for the problems deciding of whether a given argument is contained in one, resp. in all, semi-stable extensions of a given argumentation framework (the respective membership results have been shown by Dunne and Caminada [8], but matching lower bounds have been left as an open problem). We also show that stage semantics (defined in terms of conflict-free sets) are of the same complexity as semi-stable semantics. Our results therefore indicate that the considered semantics are as hard as the preferred semantics [5, 7], and thus among the most involved semantics for argumentation frameworks. In order to identify tractable subclasses, we analyze fixed-parameter tractability for the semantics under consideration in terms of tree-width and cycle-rank.

2 Background

An *argumentation framework* (AF, for short) is a pair (A, R) where A is a set of arguments and $R \subseteq A \times A$ represents the attack-relation. For an AF $F = (A, R)$, $S \subseteq A$, and $a \in A$, we call

- S *conflict-free* in F, if there are no $b, c \in S$ such that $\langle b, c\rangle \in R$,
- a *defeated* by S in F, if there is a $b \in S$ such that $\langle b, a\rangle \in R$,
- a *defended* by S in F, if for each $b \in A$ such that $\langle b, a\rangle \in R$, b is defeated by S in F,
- S *admissible* in F, if S is conflict-free in F and each $a \in S$ is defended by S in F.

To define the concepts of stage and resp. semi-stable extensions, we basically follow the conventions used in [1]. Let for an AF $F = (A, R)$ and a set $S \subseteq A$, S_R^+ be defined as $S \cup \{b \mid \exists a \in S, \text{such that } \langle a, b\rangle \in R\}$. Moreover, let us say that $S \leq_R^+ T$ holds if $S_R^+ \subseteq T_R^+$.

Definition 1. *Let $F = (A, R)$ be an AF. A set S is a* stage *(resp. a* semi-stable*) extension of F, if S is maximal conflict-free (resp. admissible) in F wrt. $\leq^+_R$.*

The following example shows that stage and semi-stable extensions are in general incomparable.

Example 1. Let $F = (\{a, b, c\}, \{\langle a, b\rangle, \langle b, c\rangle, \langle c, a\rangle\})$. Then, the only semi-stable extension of F is the empty set, while F possesses three stage extensions $\{a\}$, $\{b\}$, and $\{c\}$.

We consider the following decision problems:

- StageCred: Given AF $F = (A, R)$ and $a \in A$; is a contained in at least one stage extension of F?
- StageSkept: Given AF $F = (A, R)$ and $a \in A$; is a contained in every stage extension of F?
- SemiCred: Given AF $F = (A, R)$ and $a \in A$; is a contained in at least one semi-stable extension of F?
- SemiSkept: Given AF $F = (A, R)$ and $a \in A$; is a contained in every semi-stable extension of F?

For our forthcoming reductions, we require a particular class of quantified Boolean formulas (QBFs) which we introduce next. A $QBF^2_\forall$ formula is of the form $\forall Y \exists Z C$ where Y and Z are sets of propositional atoms from a countable domain U, and C is a collection of clauses (which we shall represent as sets) over literals built from atoms $Y \cup Z$. For a variable y, we use $\bar{y}$ to represent its negation. Moreover, $\bar{\bar{y}}$ stands for y, etc. We say that a QBF $\forall Y \exists Z C$ is true iff, for each $I_Y \subseteq Y$ there exists an $I_Z \subseteq Z$, such that for each $c \in C$,

$$\big(I_Y \cup I_Z \cup \{\bar{x} \mid x \in (Y \cup Z) \setminus (I_Y \cup I_Z)\}\big) \cap c \neq \emptyset. \tag{1}$$

Example 2. Consider the QBF

$$\Phi = \forall y_1, y_2 \exists z_3, z_4 \big\{\{y_1, y_2, z_3\},\ \{\bar{y}_2, \bar{z}_3, \bar{z}_4)\},\ \{\bar{y}_1, \bar{y}_2, z_4\}\big\}.$$

It can be checked that this QBF is true.

We recall that the problem $QSAT^2_\forall$ (i.e. given a $QBF^2_\forall$ formula Φ, decide whether Φ is true) is Π^P_2-complete.

3 Complexity of Semi-Stable and Stage Semantics

As already mentioned, we consider a countable set U of propositional atoms (in what follows, we use atoms and arguments interchangeably). Moreover, we have the following pairwise disjoint sets of arguments $\bar{U} = \{\bar{u} \mid u \in U\}$, $U' = \{u' \mid u \in U\}$, $\bar{U}' = \{\bar{u}' \mid u \in U\}$. For any set $V \subseteq U$, we use $\bar{V}$, V', $\bar{V}'$, also as renaming schemes in the usual way (for instance, V' denotes the set $\{v' \mid v \in V\}$). Finally, we use further new arguments $t, \bar{t}, b$ and $\{c_1, c_2, \ldots\}$.

We make use of the following reduction from $QBF^2_\forall$ formulas to AFs.

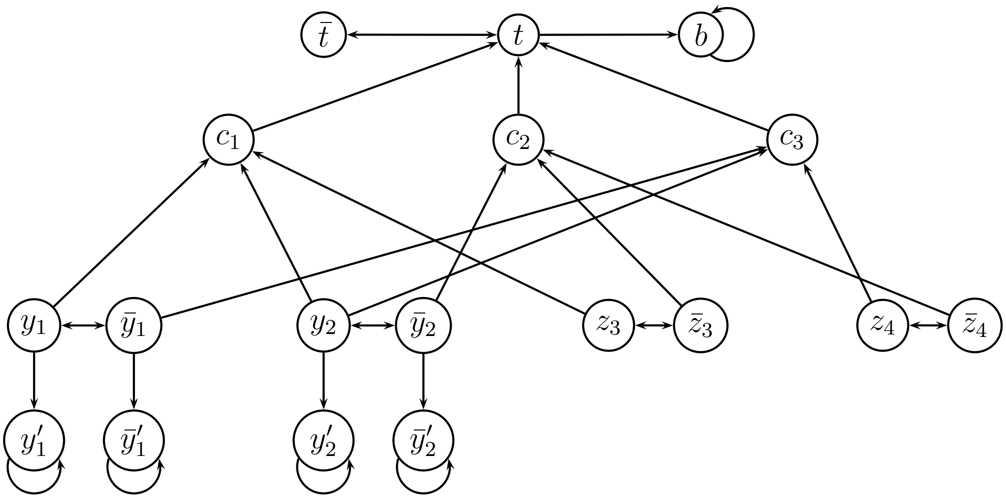

Figure 1: Argumentation framework F_Φ for Φ as given in Example 2.

Reduction 1. Given a $QBF^2_\forall$ formula $\Phi = \forall Y \exists Z C$, we define $F_\Phi = (A, R)$, where

$$
\begin{aligned}
A &= \{t, \bar{t}, b\} \cup C \cup Y \cup \bar{Y} \cup Y' \cup \bar{Y}' \cup Z \cup \bar{Z} \\
R &= \{\langle c, t\rangle \mid c \in C\} \cup \\
&\quad \{\langle x, \bar{x}\rangle, \langle \bar{x}, x\rangle \mid x \in Y \cup Z\} \cup \\
&\quad \{\langle y, y'\rangle, \langle \bar{y}, \bar{y}'\rangle, \langle y', y'\rangle, \langle \bar{y}', \bar{y}'\rangle \mid y \in Y\} \cup \\
&\quad \{\langle l, c\rangle \mid \text{literal } l \text{ occurs in } c \in C\} \cup \\
&\quad \{\langle t, \bar{t}\rangle, \langle \bar{t}, t\rangle, \langle t, b\rangle, \langle b, b\rangle\}.
\end{aligned}
$$

As an example, consider Figure 1 which illustrates the corresponding AF F_Φ for Φ from Example 2.

We start with a few basic properties, any such AF F_Φ satisfies.

Lemma 1. *For every stage (resp. semi-stable) extension S of an AF $F_\Phi = (A, R)$, the following propositions hold:*

1. $b \notin S$, *as well as* $y' \notin S$ *and* $\bar{y}' \notin S$ *for each* $y \in Y$.
2. $x \notin S \Leftrightarrow \bar{x} \in S$ *for each* $x \in \{t\} \cup Y \cup Z$.

Proof. Let $\Phi = \forall Y \exists Z C$ and $F_\Phi = (A, R)$ be the corresponding AF.

ad 1) Clear, since all these arguments are self-defeating and therefore they cannot be in a set which is conflict-free in F_Φ. Consequently, no such argument can occur in a stage or semi-stable extension of F_Φ.

ad 2) Obviously, for each $x \in \{t\} \cup Y \cup Z$, $\{x, \bar{x}\} \subseteq S$ cannot hold, since S has to be conflict-free in F_Φ. It remains to show $\{x, \bar{x}\} \cap S \neq \emptyset$. Towards a contradiction, let us assume there exists such an x, such that $\{x, \bar{x}\} \cap S = \emptyset$ holds for a stage (resp. semi-stable) extension S of F_Φ.

Let us first assume the set $T = S \cup \{\bar{x}\}$ is conflict-free in F_Φ (this is always the case for $x = t$, but not necessarily for $x \in Y \cup Z$). We have $S <^+_R T$, since $S \subset T$ and $\bar{x} \notin S^+_R$ (since $\{x, \bar{x}\} \cap S = \emptyset$). This already shows that S then cannot be a stage extension of F_Φ. We proceed with the case where we assume S to be a semi-stable extension of F_Φ. Thus, S is admissible in F_Φ. But then, T remains admissible in F_Φ ($\bar{x} \in T$ defends itself in F_Φ; each other argument in T is already defended by S in F_Φ). Hence, in this case, S would not be a semi-stable extension of F_Φ.

Let us thus assume that $S \cup \{\bar{x}\}$ is not conflict-free in F_Φ, we know $x \in Y \cup Z$, and thus there is a $c \in C$, such that $c \in S$ and $\langle \bar{x}, c\rangle \in R$. Consequently, $t \notin S$ but since we assume S to be a stage or semi-stable extension of F_Φ, we can assume $\bar{t} \in S$ (we already have shown $\{t, \bar{t}\} \cap S \neq \emptyset$). Further, as there is a $c \in C$ with $c \in S$ and $\langle \bar{x}, c\rangle \in R$, and since we have $x \notin S$, c is not defended by S in F_Φ. But then, S cannot be a semi-stable extension of F_Φ (as S is not admissible in F_Φ).

It remains to show that S is not a stage extension of F_Φ. To this end, let $T = (S \setminus \{c \in C \mid \langle \bar{x}, c\rangle \in R\}) \cup \{\bar{x}\}$. One can check that T is conflict-free in F_Φ. Moreover, we again have $S <^+_R T$. In fact, for the removed arguments $c \in C$, we have $c \in T^+_R$ (since for each such c, $\langle \bar{x}, c\rangle \in R$ and $\bar{x} \in T$); moreover, the only argument defeated by such c is t, but $t \in T^+_R$, since $\bar{t} \in T$ (recall that $\bar{t} \in S$ and thus $\bar{t} \in T$). This shows that S cannot be a stage (resp. semi-stable) extension of F_Φ and we arrive at the desired contradiction. □

Lemma 2. *Let $Y^* = Y \cup \bar{Y} \cup Y' \cup \bar{Y}'$ and S, T be conflict-free sets in $F_\Phi = (A, R)$. Then $S \cap Y^* \subseteq T \cap Y^*$ iff $(S \cap Y^*)^+_R \subseteq (T \cap Y^*)^+_R$ and further $S \cap Y^* = T \cap Y^*$ iff $(S \cap Y^*)^+_R = (T \cap Y^*)^+_R$.*

Proof. First, assume $S \cap Y^* \subseteq T \cap Y^*$. Let $l \in (S \cap Y^*)^+_R$. If $l \in S \cap Y^*$, then $l \in T \cap Y^*$ and thus $l \in (T \cap Y^*)^+_R$. Otherwise there exists a $k \in S \cap Y^*$, such that $\langle k, l\rangle \in R$ and $k \in S \cap Y^*$. By assumption $k \in T \cap Y^*$ and since $\langle k, l\rangle \in R$, $l \in (T \cap Y^*)^+_R$. So, assume now $(S \cap Y^*)^+_R \subseteq (T \cap Y^*)^+_R$ and let $l \in S \cap Y^*$. By Lemma 1(1), l is either of form y or $\bar{y}$. If $l \in S \cap Y^*$, then $l, \bar{l}, l' \in (S \cap Y^*)^+_R$. However, since S is conflict-free in F_Φ, $\bar{l} \notin S \cap Y^*$, Thus, $\bar{l}' \notin (S \cap Y^*)^+_R$. We get $l, \bar{l}, l' \in (T \cap Y^*)^+_R$ and $\bar{l}' \notin (T \cap Y^*)^+_R$. The latter implies $\bar{l} \notin (T \cap Y^*)$. But then, $l \in T \cap Y^*$ follows from $l, \bar{l} \in (T \cap Y^*)^+_R$.

This shows $S \cap Y^* \subseteq T \cap Y^*$ iff $(S \cap Y^*)^+_R \subseteq (T \cap Y^*)^+_R$. By symmetry, $S \cap Y^* = T \cap Y^*$ iff $(S \cap Y^*)^+_R = (T \cap Y^*)^+_R$ follows. □

Lemma 3. *Let Φ be a $QBF^2_\forall$ formula. If Φ is true, then t is contained in every stage and in every semi-stable extension of F_Φ.*

Proof. Suppose $\Phi = \forall Y \exists Z C$ is true and let, towards a contradiction, S be a stage or a semi-stable extension of $F_\Phi = (A, R)$ with $t \notin S$. By Lemma 1(2), we know that for each $y \in Y$, either y or $\bar{y}$ is in S. Let $I_Y = Y \cap S$. Since Φ is true we know there exists an $I_Z \subseteq Z$, such that (1) holds, for each $c \in C$. Consider now the set

$$T = I_Y \cup I_Z \cup \{\bar{x} \mid x \in (Y \cup Z) \setminus (I_Y \cup I_Z)\} \cup \{t\}.$$

We show that T is admissible in F_Φ and that $S <^+_R T$ holds. This will contradict our assumption in both cases, i.e. that S is a stage or a semi-stable extension of F_Φ. It is easily verified that T is conflict-free in F_Φ. Next we show that each $a \in T$ is defended by T in F_Φ. This is quite obvious for each $a \in T$ except t, since all those arguments defend themselves. To have t defended by T in F_Φ, each argument $c \in C$ has to be attacked by an element from T. But this is the case since (1) holds and by the construction of F_Φ, i.e. by the definition of attacks $\{\langle l, c\rangle \mid$ literal l occurs in $c\}$, each such attacker c is attacked by an argument $x \in Y \cup \bar{Y} \cup Z \cup \bar{Z}$. It remains to show $S <^+_R T$. By Lemma 2, $(S \cap Y^*)^+_R = (T \cap Y^*)^+_R$, for $Y^* = Y \cup \bar{Y} \cup Y \cup \bar{Y}'$. Moreover, by Lemma 1(2) either z or $\bar{z}$ in S, for each $z \in Z$; the same holds for T, by definition. We observe that $S^+_R \cap (Z \cup \bar{Z}) = T^+_R \cap (Z \cup \bar{Z}) = (Z \cup \bar{Z})$. Moreover, we already have argued that each $c \in C$ is attacked by some argument in T. Let $A^- = A \setminus \{t, \bar{t}, b\}$. So far, we thus have shown that $S^+_R \cap A^- \subseteq T^+_R \cap A^- = I_Y \cup I'_Y \cup (\bar{Y} \setminus \bar{I}_Y) \cup (\bar{Y}' \setminus \bar{I}'_Y) \cup Z \cup \bar{Z} \cup C$. We finally observe that $S^+_R \cap \{t, \bar{t}, b\} = \{t, \bar{t}\} \subset \{t, \bar{t}, b\} = T^+_R \cap \{t, \bar{t}, b\}$, since $t \notin S$ by assumption and $t \in T$ by definition. This shows $S <^+_R T$ as desired. □

We are now prepared to give our first main result.

Theorem 1. SemiSkept *is* Π^P_2*-hard.*

Proof. We use our reduction from $QBF^2_\forall$ formulas to AFs and show that, for each such QBF Φ, it holds that t is contained in all semi-stable extensions of F_Φ iff Φ is true. Since F_Φ can be constructed from Φ in polynomial time (and even in logarithmic space), the claim then follows. Let $\Phi = \forall Y \exists Z C$ and $F_\Phi = (A, R)$ be the corresponding AF. The if direction is captured by Lemma 3. We prove the only-if direction by showing that if Φ is false, then there exists a semi-stable extension S of F_Φ such that $t \notin S$.

In case Φ is false, there exists an $I_Y \subseteq Y$, such that for each $I_Z \subseteq Z$, there exists a $c \in C$, such that

$$\big(I_Y \cup I_Z \cup \{\bar{x} \mid x \in (Y \cup Z) \setminus (I_Y \cup I_Z)\}\big) \cap c = \emptyset. \tag{2}$$

Consider now a maximal (wrt. $\leq^+_R$) admissible (in F_Φ) set S, such that $I_Y \subseteq S$ (note that such a set exists, since I_Y itself is admissible in F_Φ). Using Lemma 2, one can show that S then has to be a semi-stable extension of F_Φ. To wit, let T be an admissible (in F_Φ) set such that $I_Y \not\subseteq T$. By Lemma 2 it holds that $(S \cap Y^*)^+_R \not\subseteq (T \cap Y^*)^+_R$ and therefore $S^+_R \not\subseteq T^+_R$. Putting this together with the maximality of S in the set $\{T \mid T$ is admissible in F_Φ and $I_Y \subseteq T\}$ we get that there is no admissible (in F_Φ) set T, such that $S^+_R \subset T^+_R$. Hence, S is a semi-stable extension of F_Φ.

It remains to show $t \notin S$. We prove this by contradiction and assume $t \in S$. As S is admissible in F_Φ, S defends t and therefore it defeats all $c \in C$. As all attacks against arguments in C come from $Y \cup \bar{Y} \cup Z \cup \bar{Z}$, the set $U = \big(I_Y \cup (S \cap (Z \cup \bar{Z})) \cup \{\bar{y} \mid y \in Y \setminus I_Y\}\big)$ defeats all $c \in C$. By Lemma 1(2), for each $z \in Z$, either z or $\bar{z}$ is contained in S. We get an equivalent characterization for U by $U = \big(I_Y \cup I_Z \cup \{\bar{x} \mid x \in (Y \cup Z) \setminus (I_Y \cap I_Z)\}\big)$ with $I_Z = S \cap Z$. Thus, for all $c \in C$,

$$\big(I_Y \cup I_Z \cup \{\bar{x} \mid x \in (Y \cup Z) \setminus (I_Y \cup I_Z)\}\big) \cap c \neq \emptyset.$$

which contradicts assumption (2). □

Theorem 2. SemiCred *is* Σ^P_2*-hard.*

Proof. In the proof of Theorem 1, we have shown that a $QBF_\forall^2$ formula Φ is true iff t is contained in each semi-stable extension of F_Φ. According to Lemma 1(2), this holds iff $\bar{t}$ is not contained in any semi-stable extension of F_Φ. Thus, the complementary problem of $\mathsf{SemiCred}$ is also Π_2^P-hard. Σ_2^P-hardness of $\mathsf{SemiCred}$ follows immediately. □

We now turn our attention to the stage semantics.

Theorem 3. $\mathsf{StageSkept}$ *is* Π_2^P*-hard.*

Proof. We again use our reduction from $QBF_\forall^2$ formulas to AFs and show that, for each such QBF Φ, it holds that t is contained in all stage extensions of F_Φ iff Φ is true. Thus, let $\Phi = \forall Y \exists Z C$ and $F_\Phi = (A, R)$ be the corresponding AF. The if direction is already captured by Lemma 3. We prove the only-if direction by showing that, if Φ is false, then there exists a stage extension S of F_Φ such that $t \notin S$.

If Φ is false, there exists an $I_Y \subseteq Y$, such that for each $I_Z \subseteq Z$, we have a $c \in C$ with

$$\big(I_Y \cup I_Z \cup \{\bar{x} \mid x \in (Y \cup Z) \setminus (I_Y \cup I_Z)\}\big) \cap c = \emptyset. \tag{3}$$

Consider the collection $W = \{S \mid I_Y \subseteq S, S \text{ is conflict-free in } F_\Phi\}$ of conflict-free sets in F_Φ. Using Lemma 2, we can show that for every conflict-free (in F_Φ) set T, $S \leq_R^+ T$ implies $I_Y \subseteq T$. For verifying $\leq_R^+$-maximality of a set $S \in W$ we thus can restrict ourselves to sets $T \in W$.

It remains to show that there is a stage extension S in W with $t \notin S$. We prove that (i) for every set $S \in W$ with $t \in S$, there exists a $c \in C$, such $c \notin S_R^+$; and (ii) existence of a set $S \in W$ such that $C \subseteq S_R^+$. Note that (i)+(ii) imply existence of a stage extension S of F_Φ with $t \notin S$.

We prove (i) by contradiction and assume that $C \subseteq S_R^+$. As S is conflict-free in F_Φ and $t \in S$, we get $C \cap S = \emptyset$. Since $C \subseteq S_R^+$, S defeats all $c \in C$. As all attacks against C come from $Y \cup \bar{Y} \cup Z \cup \bar{Z}$, the set $U = \big(I_Y \cup (S \cap (Z \cup Z')) \cup \{\bar{y} \mid y \in Y \setminus I_Y\}\big)$ defeats all $c \in C$. By Lemma 1(2), for each $z \in Z$, either z or $\bar{z}$ is contained in S and so we get $U = \big(I_Y \cup I_Z \cup \{\bar{x} \mid x \in (Y \cup Z) \setminus (I_Y \cup I_Z)\}\big)$ with $I_Z = S \cap Z$. Thus, for each $c \in C$, $U \cap c \neq \emptyset$, which contradicts assumption (3).

To show (ii) we just construct such a set $S = U \cup V$ using

$$\begin{aligned} U &= I_Y \cup \{\bar{y} \in Y \setminus I_Y\} \cup Z \\ V &= \{c \in C \mid \nexists u \in U \text{ with } \langle u, c\rangle \in R\}. \end{aligned}$$

It is easy to verify that S is conflict-free in F_Φ. It remains to show that for all $c \in C$, $c \in S_R^+$ holds. Note that for each $c \in C$ we have that either c is attacked by U or contained in V. In both cases, $c \in S_R^+$ is clear. □

The following result is proven analogously to Theorem 2.

Theorem 4. $\mathsf{StageCred}$ *is* Σ_2^P*-hard.*

Our hardness results can be extended to AFs without self-defeating arguments. To this end, we adapt our reduction by replacing all self-defeating arguments in the framework F_Φ by cycles of odd length (for instance, of length 3). Figure 2 illustrates such a framework F_Φ^m for our example QBF.

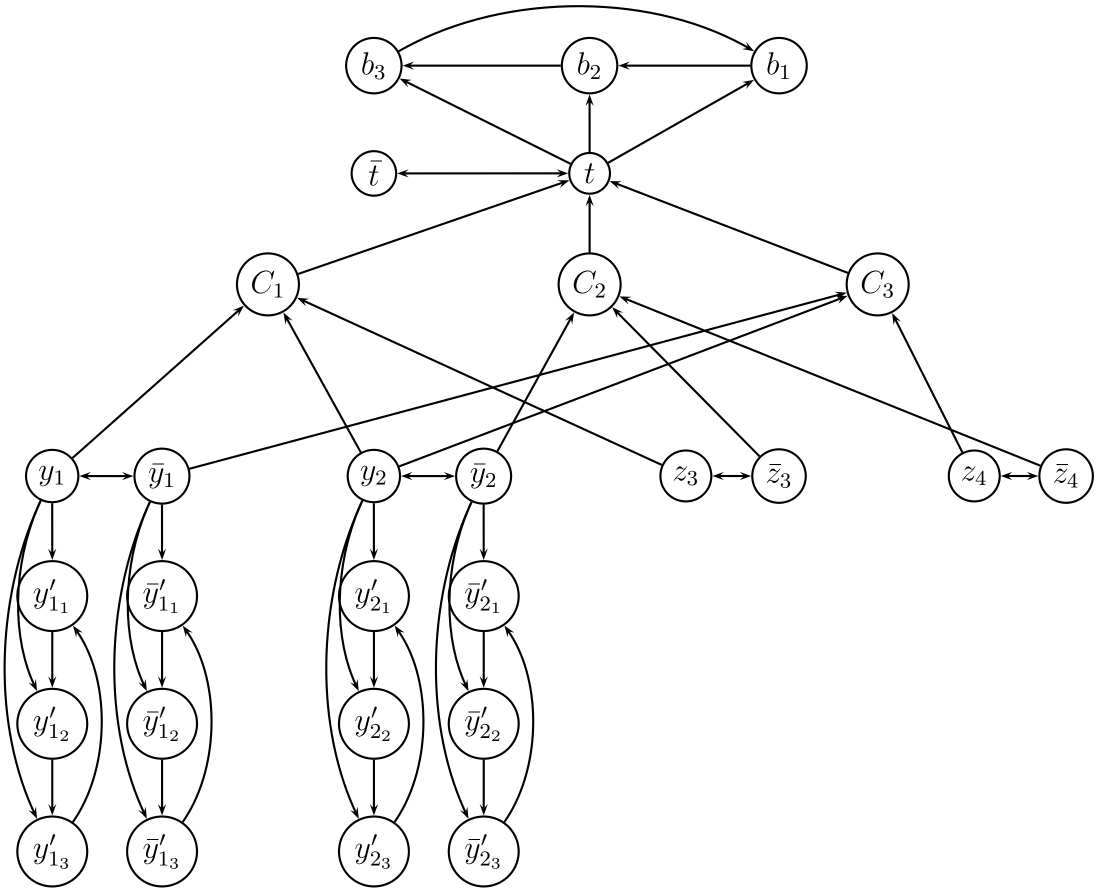

Figure 2: The modified framework F^m_Φ for Φ from Example 2.

In case of semi-stable extensions, we use the fact that the only admissible set of an odd length cycle is the empty set. We thus get that a set S is a semi-stable extension of F_Φ iff S is a semi-stable extension of F^m_Φ.

The same construction can be used for stage semantics, although the argumentation is slightly different: As stage extensions only require conflict-freeness and not admissibility, the arguments of the introduced cycles may now be part of stage extensions. However, to repair the correctness proofs for the modified reduction, we use the observation that for each cycle of length 3 at most one argument can be in a stage extension S (see also Example 1) and at least one argument in the cycle is not defeated by S. Thus each such cycle contributes in three different but incomparable ways to stage extensions. More formally, let let A^m be the set of arguments in F^m_Φ, $X = \{b\} \cup Y' \cup \bar{Y}'$ and denote by x^- be the (unique) attacker of an argument $x \in X$ in the original framework $F_\Phi = (A, R)$. Then, we get that (i) if S is a stage extension of F_Φ, then each $S' \subseteq A^m$, such that $S' \cap A = S$ and for each $x \in X$,

$$card(S' \cap \{x_1, x_2, x_3\}) = \begin{cases} 1 & \text{if } x^- \notin S \\ 0 & \text{otherwise} \end{cases}$$

is a stage extension of F^m_Φ; and (ii) if S is a stage extension of F^m_Φ, then $S \cap A$ is a stage extension of F_Φ (where A is the set of arguments in F_Φ). This correspondence between stage extensions suffices to show that our hardness results carry over to self-defeat free AFs.

We summarize our results in terms of completeness results. The matching upper bounds for semi-stable semantics have been reported in [8]; for the stage semantics we give them in the proof of the following theorem.

Theorem 5. *Problems* StageCred *and* SemiCred *are* Σ_2^P*-complete;* StageSkept *and* SemiSkept *are* Π_2^P*-complete. For all problems, hardness holds even for AFs without self-defeating arguments.*

Proof. Hardness is by Theorems 1–4 and by the observations in the previous paragraphs.

For the matching upper bounds, we first consider the following problem which we show to be in coNP: Given an AF $F = (A, R)$ and a set $S \subseteq A$, is S a stage (resp. a semi-stable) extension of F. Let $cf(F)$ denote the collection of conflict-free sets $S \subseteq A$ of F and $adm(F)$ denote the collection of sets $S \subseteq A$, admissible in F. By definition, S is a stage (resp. a semi-stable) extension of F iff (i) $S \in \sigma(F)$ and (ii) $\forall\, T \subseteq A$, $T \in \sigma(F)$ only if $S_R^+ \not\subset T_R^+$, for $\sigma = cf$ (resp. $\sigma = adm$). Given S, we can decide $S \in \sigma(F)$ in polynomial time, for $\sigma \in \{cf, adm\}$. For the complement of (ii), we guess a set T and then we verify (again, in polynomial time), whether $T \in \sigma(F)$, for $\sigma \in \{cf, adm\}$. This yields membership in NP for the complement of (ii), thus, given set S, (ii) is in coNP, and thus the entire problem is in coNP.[1]

We now can give algorithms for StageCred and SemiCred as follows. We have given an AF $F = (A, R)$ and an argument $a \in A$. We guess a set $S \subseteq A$ with $a \in S$ and then use an NP-oracle (we recall that oracle calls are closed under complement), to check whether S is a stage (resp. semi-stable) extension of F. Obviously this algorithm correctly decides the considered problems. Hence, these problems are in Σ_2^P.

For StageSkept and SemiSkept we argue as follows: Given an AF $F = (A, R)$, to decide if an argument $a \in A$ is contained in each stage (resp. semi-stable) extension of F, we have to prove that every set S with $a \notin S$ is *not* a stage (resp. semi-stable) extension of F. Thus, for the complementary problem, we can guess a set S with $a \notin S$ and check whether S is a stage (resp. semi-stable) extension of F. Again, this check can be done with a single call to an NP-oracle, and thus the complementary problems are in Σ_2^P. Π_2^P-membership of StageSkept and SemiSkept follows immediately. ☐

4 Fixed Parameter Tractability

As we have shown in the previous section, all considered problems are highly intractable. A natural task is now to identify tractable subclasses of the problems. We focus on particular graph parameters and check whether bounding such parameters leads to the desired tractable fragments.

4.1 Tree-Width

One such parameter for graph problems is tree-width [13]. Intuitively, the tree-width of a graph measures the tree-likeness of the graph.

[1] For semi-stable semantics, this problem is also coNP-complete, cf. [8].

Definition 2. *Let $G = (V, E)$ be an undirected graph. A* tree decomposition *of G is a pair $\langle \mathcal{T}, \mathcal{X} \rangle$ where $\mathcal{T} = \langle V_\mathcal{T}, E_\mathcal{T} \rangle$ is a tree and $\mathcal{X} = (X_t)_{t \in V_\mathcal{T}}$ such that:*

1. *$\bigcup_{t \in V_\mathcal{T}} X_t = V$, i.e. $\mathcal{X}$ is a cover of V,*
2. *for each $v \in V$ the subgraph of $\mathcal{T}$ induced by $\{t \mid v \in X_t\}$ is connected,*
3. *for each edge $\{v_i, v_j\} \in E$ there exists an X_t with $\{v_i, v_j\} \subseteq X_t$.*

The width *of a decomposition $\langle \mathcal{T}, \mathcal{X} \rangle$ is given by $\max\{|X_t| : t \in V_\mathcal{T}\} - 1$. The* tree-width *of a graph G is the minimum width over all tree decompositions of G.*

Many graph properties can be defined by formulas of monadic second-order logic (MSOL) and by Courcelle's Theorem [4] such graph properties can be efficiently decided on graphs with bounded tree-width.

Theorem 6. *Let be K a class of graphs for which the tree-width is bounded by some constant k and Π be a MSOL-definable property. For each such $G \in K$, $G \in \Pi$ is decidable in linear time wrt. the size of G.*

With Courcelle's theorem at hand, we have a powerful tool to classify graph problems as fixed-parameter tractable. In order to apply the concept of tree-width to argumentation frameworks, we define the decision problems $\mathsf{SemiSkept}_k^{tw}$, $\mathsf{SemiCred}_k^{tw}$, $\mathsf{StageSkept}_k^{tw}$ and $\mathsf{StageCred}_k^{tw}$ in the same way as SemiSkept, SemiCred, StageSkept and StageCred, but restricted to AFs, which, when interpreted as undirected graphs, have tree-width $\leq k$.

Theorem 7. *For fixed k, the problems $\mathsf{SemiSkept}_k^{tw}$, $\mathsf{SemiCred}_k^{tw}$, $\mathsf{StageSkept}_k^{tw}$ and $\mathsf{StageCred}_k^{tw}$ are decidable in linear time.*

Proof. Let us consider the following building blocks in MSOL:

$$
\begin{aligned}
U \subseteq_R^+ V &= \forall x \Big(\big(x \in U \vee \exists y (y \in U \wedge \langle y, x \rangle \in R) \big) \rightarrow \\
&\qquad \big(x \in V \vee \exists y (y \in V \wedge \langle y, x \rangle \in R) \big) \Big) \\
U \subset_R^+ V &= U \subseteq_R^+ V \wedge \neg (V \subseteq_R^+ U) \\
\mathrm{cf}_R(U) &= \forall x, y \big(\langle x, y \rangle \in R \rightarrow (\neg x \in U \vee \neg y \in U) \big) \\
\mathrm{adm}_R(U) &= \mathrm{cf}_R(U) \wedge \forall x, y \Big((\langle x, y \rangle \in R \wedge y \in U) \rightarrow \exists z (z \in U \wedge \langle z, x \rangle \in R) \Big) \\
\mathrm{semi}_{(A,R)}(U) &= \mathrm{adm}_R(U) \wedge \neg \exists V (V \subseteq A \wedge \mathrm{adm}_R(V) \wedge U \subset_R^+ V) \\
\mathrm{stage}_{(A,R)}(U) &= \mathrm{cf}_R(U) \wedge \neg \exists V (V \subseteq A \wedge \mathrm{cf}_R(V) \wedge U \subset_R^+ V)
\end{aligned}
$$

One can show that $\mathrm{semi}_F(U)$ characterizes the semi-stable extensions of an AF F, while $\mathrm{stage}_F(U)$ characterizes the stage extensions of F. The required checks for the considered problems are easily added to these formulas. Theorem 6 then yields the desired result. □

4.2 Directed Graph Measures

As argumentation frameworks are directed graphs it seems natural to consider directed graph measures to get larger tractable fragments than those we capture with bounded (undirected) tree-width. Unfortunately, it turns out that the considered problems remain hard when bounding typical directed graph measures. We illustrate this fact by using cycle rank [9] as a parameter.

Definition 3. *Let $G = \langle V, E \rangle$ be a directed graph. The cycle rank of G, $cr(G)$, is defined as follows: An acyclic graph has $cr(G) = 0$. If G is strongly connected then $cr(G) = 1 + min_{v \in V_G} cr(G \setminus v)$. Otherwise, $cr(G)$ is the maximum cycle rank among all strongly connected components of G.*

Theorem 8. *The problems* SemiSkept, StageSkept *(resp.* SemiCred, StageCred*) remain Π_2^P-hard (resp. Σ_2^P-hard), even if restricted to AFs which have a cycle-rank of* 1.

Proof. Its easy to see that every framework of the form F_Φ has cycle-rank 1 and therefore we have an reduction from $QBF_\forall^2$ formulas to an AF with cycle-rank 1. In fact, the strongly connected components of F_Φ are

$$\mathrm{SCC}(F_\Phi) = \{\{y_i, \bar{y}_i\}, \{z_i, \bar{z}_i\}, \{t, \bar{t}\}, \{y_i'\}, \{\bar{y}_i'\}, \{z_i'\}, \{\bar{z}_i'\}, \{b\}\}.$$

As each of these components can be made acyclic by removing one vertex, the cycle-rank of F_Φ is thus 1. □

By results in [12, 11, 10] it follows that a problem which is hard for bounded cycle-rank remains hard for bounding other directed graph measures, i.e. directed path-width, Kelly-width, DAG-width and directed tree-width.

5 Conclusion

In this note, we provided novel complexity results for abstract argumentation frameworks in terms of sceptical and resp. credulous acceptance under semi-stable and stage semantics (as defined in [1]). In case of the semi-stable semantics, we improved the existing $P_{||}^{NP}$-lower bound [8] to hardness for classes Π_2^P (resp. Σ_2^P). Together with existing upper bounds, we thus obtained completeness for complexity classes on the second level of the polynomial hierarchy, answering an open question raised by Caminada and Dunne [8]. Furthermore, we showed that stage semantics leads to the same complexity. To the best of our knowledge, no complexity results for this semantics have been obtained so far. Finally, we gave some results in terms of bounding some problem parameter. As a positive result, we could show that bounded tree-width leads to tractable subclasses of the problems under consideration. As a negative result, we gave evidence that more natural parameters for argumentation frameworks, as e.g. cycle-rank, are not applicable to fixed-parameter tractability results.

Acknowledgments

This work was supported by the Vienna Science and Technology Fund (WWTF) under grant ICT08-028.

References

[1] Pietro Baroni and Massimiliano Giacomin. Semantics of abstract argument systems. In I. Rahwan and G. Simari, editors, *Argumentation in Artificial Intelligence*, pages 25–44. Springer, 2009.

[2] Trevor J. M. Bench-Capon and Paul E. Dunne. Argumentation in artificial intelligence. *Artif. Intell.*, 171(10-15):619–641, 2007.

[3] Martin Caminada. Semi-stable semantics. In Paul E. Dunne and Trevor J. M. Bench-Capon, editors, *Proceedings of the 1st Conference on Computational Models of Argument* (*COMMA 2006*), volume 144 of *Frontiers in Artificial Intelligence and Applications*, pages 121–130. IOS Press, 2006.

[4] Bruno Courcelle. Graph rewriting: An algebraic and logic approach. In *Handbook of Theoretical Computer Science, Volume B: Formal Models and Sematics (B)*, pages 193–242. 1990.

[5] Yannis Dimopoulos and Alberto Torres. Graph theoretical structures in logic programs and default theories. *Theor. Comput. Sci.*, 170(1-2):209–244, 1996.

[6] Phan Minh Dung. On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games. *Artif. Intell.*, 77(2):321–358, 1995.

[7] Paul E. Dunne and Trevor J. M. Bench-Capon. Coherence in finite argument systems. *Artif. Intell.*, 141(1/2):187–203, 2002.

[8] Paul E. Dunne and Martin Caminada. Computational complexity of semi-stable semantics in abstract argumentation frameworks. In Steffen Hölldobler, Carsten Lutz, and Heinrich Wansing, editors, *Proceedings of the 11th European Conference on Logics in Artificial Intelligence* (*JELIA 2008*), volume 5293 of *LNCS*, pages 153–165. Springer, 2008.

[9] Lawrence C. Eggan. Transition graphs and the star height of regular events. *Michigan Math. J.*, 10:385–397, 1963.

[10] Hermann Gruber. Digraph complexity measures and applications in formal language theory. *Proceedings of the 4th Workshop on Mathematical and Engineering Methods in Computer Science (MEMICS 2008)*, pages 60–67, 2008.

[11] Paul Hunter and Stephan Kreutzer. Digraph measures: Kelly decompositions, games, and orderings. *Theor. Comput. Sci.*, 399(3):206–219, 2008.

[12] Paul William Hunter. Complexity and infinite games on finite graphs. Technical Report UCAM-CL-TR-704, University of Cambridge, Computer Laboratory, November 2007.

[13] Neil Robertson and Paul D. Seymour. Graph minors. III. planar tree-width. *J. Comb. Theory, Ser. B*, 36(1):49–64, 1984.

[14] Bart Verheiji. Two approaches to dialectical argumentation: Admissible sets and argumentation stages. In J. Meyer and L. van der Gaag, editors, *Proceedings of the 8th Dutch Conference on Artificial Intelligence (NAIC'96)*, pages 357–368, 1996.